

Courant algebroids, derived brackets and even symplectic supermanifolds

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Abstract

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In this dissertation we study Courant algebroids, objects that first appeared in the work of T. Courant on Dirac structures; they were later studied by Liu, Weinstein and Xu who used Courant algebroids to generalize the notion of the Drinfeld double to Lie bialgebroids. As a first step towards understanding the complicated properties of Courant algebroids, we interpret them by associating to each Courant algebroid a strongly homotopy Lie algebra in a natural way.

Next, we propose an alternative construction of the double of a Lie bialgebroid as a homological hamiltonian vector field on an even symplectic supermanifold. The classical BRST complex and the Weil algebra arise as special cases. We recover the Courant algebroid via the derived bracket construction and give a simple proof of the doubling theorem of Liu, Weinstein and Xu. We also introduce a generalization, quasi-Lie bialgebroids, analogous to Drinfeld's quasi-Lie bialgebras; we show that the derived bracket construction in this case also yields a Courant algebroid.

Finally, we compute the Poisson cohomology of a one-parameter family of $SU(2)$ -covariant Poisson structures on S^2 . As an application, we show that these structures are non-trivial deformations of each other, and that they do not admit rescaling.

Professor Alan D. Weinstein
Dissertation Committee Chair

To the memory of Nikolai Afanasievich Pravdin.

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Chapter 1

Introduction

The first example of a Courant algebroid appeared in the work of T. Courant [11] on Dirac structures. These structures are a simultaneous generalization of pre-symplectic and Poisson structures; they appear in Dirac's theory of constrained mechanical systems. A *Dirac structure* on a manifold M is a subbundle $L \subset TM \oplus T^*M$ that is maximally isotropic with respect to the canonical symmetric bilinear form on $TM \oplus T^*M$, and which satisfies a certain integrability condition. To formulate the integrability condition, Courant introduced a bilinear skew-symmetric bracket operation

$$[X + \xi, Y + \eta] = [X, Y] + (L_X\eta - L_Y\xi + \frac{1}{2}d(i_Y\xi - i_X\eta))$$

on sections of $TM \oplus T^*M$; the condition is that the sections of L be closed under this bracket. As one can see, the Courant bracket is completely natural, in the sense that it does not depend on any additional structure for its definition, but it has rather complicated properties. In particular, it does not satisfy the Leibniz rule with respect to multiplication by functions or the Jacobi identity. The “defects” in both cases are differentials of certain expressions depending on the bracket and the bilinear form; hence they disappear upon restriction to a Dirac subbundle. A Dirac subbundle transverse to T^*M is the graph of a 2-form ω , whereas one transverse to TM is the graph of a bivector field π ; the integrability condition in this case reduces to the familiar $d\omega = 0$ (resp. $[\pi, \pi] = 0$). Dirac structures, as well as the Courant bracket above, were generalized in the context of formal variational calculus by Dorfman [12].

The nature of the Courant bracket itself remained unclear until several years later when it was observed by Liu, Weinstein and Xu [31] that $TM \oplus T^*M$ endowed with the

Courant bracket plays the role of a “double” object, in the sense of Drinfeld [13], for a pair of Lie algebroids over M . *Lie algebroids* are structures on vector bundles that combine the features of both Lie algebras and the tangent bundle, and include foliations, Poisson manifolds, Lie group actions, Dirac structures and principal bundles as special cases. Many differential-geometric and Lie-theoretic constructions carry over to Lie algebroids. For example, a pair of Lie algebras on dual vector spaces is called a *Lie bialgebra* if a certain compatibility condition between them is satisfied. Lie bialgebras are linearizations of Poisson-Lie groups and semi-classical limits of quantum groups; they also provide a tool for generating classical integrable systems [10] [13]. Likewise, one defines a *Lie bialgebroid* to be a pair of Lie algebroid structures on dual vector bundles satisfying a compatibility condition. Lie bialgebroids were first introduced by Mackenzie and Xu [35] as linearizations of Poisson groupoids. Examples of Lie bialgebroids for which neither of the Lie algebroid structures is trivial include Lie bialgebras, Poisson manifolds and Poisson-Nijenhuis manifolds [26]; Lie bialgebroids were recently found to be the geometric structure behind the Classical Dynamical Yang-Baxter equation [15] [6].

A very useful tool for studying Lie bialgebras is the *Drinfeld double*, which is the Lie algebra structure on the direct sum of the two dual Lie algebras constituting the bialgebra, uniquely characterized by the requirement that the two Lie algebras be subalgebras and that the canonical inner product be ad-invariant. In fact, to find Lie bialgebras, one looks for so-called *Manin triples*: a Lie algebra with an invariant inner product, together with a pair of complementary isotropic subalgebras. Unfortunately, when one tries to construct a Drinfeld double for a Lie bialgebroid, it quickly becomes clear that it cannot be a Lie algebroid if it is to satisfy the characterizing property of the double. Instead, given a pair (A, A^*) of Lie algebroids in duality, Liu, Weinstein and Xu [31] build a skew-symmetric bracket on sections of the direct sum $A \oplus A^*$ similar to the Courant bracket above. Then, they prove that if (A, A^*) is a Lie bialgebroid, $A \oplus A^*$ becomes a *Courant algebroid*, a notion they define by emulating the properties of the original Courant bracket; conversely, they show that any Courant algebroid which admits a pair of transverse Dirac subbundles (maximally isotropic subbundles whose sections are closed under the bracket) is of this form, thus extending the theory of Manin triples to Lie bialgebroids.

In this dissertation we solve several of the problems posed in [31]. First, the properties of a Courant algebroid are rather complicated; in particular, there are anomalies in the Jacobi identity and the Leibniz rule. We show that a Courant algebroid is a resolution

of a Lie algebra. It is known [7] that resolutions of Lie algebras inherit the structure of a *strongly homotopy Lie algebra*, also known as an L_∞ -algebra [28], though in a non-canonical way. We construct an L_∞ -algebra explicitly out of the Courant algebroid; the anomalies then appear as the structure identities. This work appeared in [37].

Next, it turns out that one can twist the bracket in a Courant algebroid by adding a symmetric term. The new operation, which we denote by \circ is, in general, not skew-symmetric but all the anomalies disappear. This was conjectured in [31], and we supply a proof. Sacrificing skew-symmetry has proved worthwhile: the equivalent definition of a Courant algebroid we get is not only much nicer than the old one, but also more natural, as it turns out. The Jacobi identity in the non skew-symmetric setting looks rather like a Leibniz rule: it says that $a \circ \cdot$ is a derivation of \circ . Such structures were studied by Loday, under the name of *Leibniz algebras* [32], and by Kosmann-Schwarzbach [25], under the name of *Loday algebras*. After the modification, the original Courant bracket becomes

$$(X + \xi) \circ (Y + \eta) = [X, Y] + (L_X \eta - i_Y d\xi).$$

This is the form used by Dorfman in [12]. Very recently, Ševera [43] showed that this Courant algebroid provides a natural geometric framework for studying the symmetries of two-dimensional variational problems. We use the new definition in all that follows.

Next, in what we regard as the most important part of this work, we develop an alternative approach to the construction of a Drinfeld double for Lie bialgebroids. It is based on viewing Lie bialgebroids as homological vector fields on supermanifolds. To each pair of Lie algebroids in duality we associate a pair of odd self-commuting hamiltonian functions on an even symplectic supermanifold (in fact, a cotangent bundle) and prove that the compatibility condition for a Lie bialgebroid is equivalent to the vanishing of the Poisson bracket of these two hamiltonians. The hamiltonian vector field of the sum is then homological, and we propose to call this sum the Drinfeld double. This approach was suggested by the work of Kosmann-Schwarzbach [22] who carried it out for Lie bialgebras in a purely algebraic language, without mentioning supermanifolds. However, supermanifolds provide a natural framework even in this case; moreover, the general case cannot be reduced to pure algebra or “classical” geometry, so supermanifolds are unavoidable.

The advantage of this approach is its clarity and simplicity. Moreover, several well-known objects in homological algebra arise in this setting. Thus, applying this construction to the action Lie algebroid associated to a Lie algebra action on a manifold, we get the

classical BRST complex [27], whereas applying it to the Lie bialgebroid associated to the canonical linear Poisson structure on the dual of a Lie algebra yields the Weil algebra [4]. So far as we know, this is the only “geometric” construction of the Weil algebra to date.

To recover the Courant algebroid of Liu, Weinstein and Xu, we use the *derived bracket construction* of Kosmann-Schwarzbach [25]: starting with a differential Leibniz (in particular, Lie) superalgebra, it generates a new Leibniz superalgebra of the opposite parity. In particular, Poisson and Schouten brackets arise in this way. That Courant algebroids may also arise in this way was first suggested by Kosmann-Schwarzbach, who showed, in a private discussion with the author, that if one considers the differential Lie superalgebra generated by exterior multiplications by 1-forms, contractions by vector fields and the de Rham differential, the derived bracket one gets is the original Courant bracket. What we do here is a “semiclassical” version of this, for an arbitrary Lie bialgebroid. The Lie superalgebra structure is given by the Poisson bracket on the even symplectic supermanifold, and the differential is the homological hamiltonian vector field, the Drinfeld double. The derived bracket we get is precisely the (non skew-symmetric) Courant bracket of [31]. This enables us to give a very simple proof of the doubling theorem of Liu, Weinstein and Xu mentioned above.¹

Furthermore, using this approach we are also able to generalize the notion of a *quasi-Lie bialgebra*, introduced by Drinfeld [14] and studied by Kosmann-Schwarzbach [22], to the Lie algebroid setting simply by adding cubic terms to our hamiltonian, thus answering another question posed in [31]. This also gives a Courant algebroid via the derived bracket construction, thus answering in the affirmative the question of the existence of nontrivial Courant algebroids which do not come from Lie bialgebroids. As a special case, we look at exact Courant algebroids recently classified by Ševera [43]. The cubic term in this case is just the closed 3-form whose cohomology class is the characteristic class of the Courant algebroid.

This thesis is organized as follows. In Chapter 2 we recall the notions of Lie bialgebra, Lie bialgebroid and Courant algebroid and prove that Courant algebroids can be considered as strongly homotopy Lie algebras; we then give a new definition of a Courant algebroid based on the non skew-symmetric operation and prove its equivalence to the old

¹ When this research was carried out, we learned that the picture of Lie bialgebroids as a pair of Poisson-commuting hamiltonians on a symplectic supermanifold was also considered by A. Vaintrob who studied representations of Lie algebroids; however, the relation with the Courant algebroids was not elucidated. Our work is completely independent of his.

one.

In Chapter 3 we develop the theory of Lie bialgebroids and quasi-bialgebroids in terms of even symplectic supermanifolds, give the derived bracket construction of the Courant algebroid and re-prove the doubling theorem of Liu, Weinstein and Xu, generalizing it also for quasi-bialgebroids.

In the final Chapter 4, somewhat disjoint from the rest, we study a one-parameter family of Poisson structures on S^2 covariant with respect to the action of $SU(2)$ with its standard Poisson-Lie group structure. We compute the Poisson cohomology of these structures and show, as an application, that they do not admit rescaling, and also that they are non-trivial deformations of each other.

Throughout this dissertation, a manifold will always mean a smooth real manifold, and all vector spaces, algebras, etc. are over the field of real numbers, unless otherwise specified. The Einstein summation convention is used consistently.

Chapter 2

Courant algebroids and strongly homotopy Lie algebras

In this chapter we recall the definition of a Courant algebroid first given in [31] and some of the results obtained therein. We then make the first step toward explaining the anomalies of Courant algebroids by showing that they can be considered as strongly homotopy Lie algebras. This is essentially the content of [37]. In the last section we propose an equivalent definition of a Courant algebroid which has the advantage of being anomaly-free (except for lack of skew-symmetry), and will be useful in what follows. To begin, we recall the notions of a Lie bialgebra, Lie algebroid and bialgebroid and give some examples.

2.1 Lie bialgebras

Definition 2.1.1. A *Lie bialgebra* is a vector space \mathfrak{g} together with a bilinear skew-symmetric map $\mu = [\cdot, \cdot] : \wedge^2 \mathfrak{g} \rightarrow \mathfrak{g}$ (the bracket) and a linear map $\gamma : \mathfrak{g} \rightarrow \wedge^2 \mathfrak{g}$ (the cobracket) such that the following properties are satisfied:

- \mathfrak{g} together with $[\cdot, \cdot]$ is a Lie algebra;
- \mathfrak{g}^* together with $[\cdot, \cdot]_* = \gamma^* : \wedge^2 \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ is a Lie algebra;
- γ is a 1-cocycle on the Lie algebra (\mathfrak{g}, μ) with values in the (exterior square of the) adjoint module $\wedge^2 \mathfrak{g}$, i.e.

$$\gamma([a, b]) = ad_a \gamma(b) - ad_b \gamma(a)$$

holds for all $a, b \in \mathfrak{g}$.

One sometimes calls the pair $(\mathfrak{g}, \mathfrak{g}^*)$ a Lie bialgebra with the underlying structures implicitly understood. Lie bialgebras are the infinitesimal objects corresponding to *Poisson-Lie groups* (see Appendix); they are also the semi-classical limits of *quantum groups* (see [10] for a thorough treatment and numerous examples).

Definition 2.1.2. Given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, its *double* (or *Drinfeld double*) is the vector space direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$ together with the bracket

$$[X + \xi, Y + \eta] = ([X, Y] + ad_\xi^* Y - ad_\eta^* X) + (ad_X^* \eta - ad_Y^* \xi + [\xi, \eta]_*) \quad (2.1)$$

This bracket is completely characterized by the property that both \mathfrak{g} and \mathfrak{g}^* be subalgebras of $\mathfrak{g} \oplus \mathfrak{g}^*$ and that the canonical inner product

$$\langle X + \xi, Y + \eta \rangle = \xi(Y) + \eta(X) \quad (2.2)$$

be ad -invariant; it satisfies the Jacobi identity if $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra. In fact, the notion of a Lie bialgebra is equivalent to that of a *Manin triple* which is a triple $(\mathfrak{p}, \mathfrak{p}_+, \mathfrak{p}_-)$, where \mathfrak{p} is a Lie algebra with an invariant symmetric bilinear form, and \mathfrak{p}_+ and \mathfrak{p}_- are complementary isotropic subalgebras. Manin triples abound in nature: for example, every complex semisimple Lie algebra gives rise to a Manin triple via the Iwasawa decomposition (see [33]).

2.2 Lie algebroids and bialgebroids

Definition 2.2.1. A *Lie algebroid* is a vector bundle $A \rightarrow M$ together with a Lie algebra bracket $[\cdot, \cdot]_A$ on the space of sections $\Gamma(A)$ and a bundle map $a : A \rightarrow TM$, called the *anchor*, satisfying the following conditions:

1. For any $X, Y \in \Gamma(A)$, $a[X, Y]_A = [aX, aY]$
2. For any $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$, $[X, fY]_A = f[X, Y]_A + (a(X)f)Y$

In other words, the sections of the bundle act on smooth functions by derivations via the anchor in such a way that brackets act as commutators, and the behavior of the bracket with respect to multiplication by functions is governed by the Leibniz rule. Thus, Lie algebroids are a straightforward generalization of the tangent bundle. They are also the

infinitesimal objects corresponding to Lie groupoids [34]; when the base manifold is a point, a Lie groupoid reduces to a Lie group, while a Lie algebroid is just a Lie algebra.

A Lie algebroid structure on $A \rightarrow M$ gives rise to the following structures, dual to one another. The *generalized Schouten bracket* is defined as the unique extension $[\cdot, \cdot]_A$ of the Lie bracket on $\Gamma(A)$ and the action of $\Gamma(A)$ on functions to $\Gamma(\bigwedge^* A)$ such that:

1. $[X, Y]_A = -(-1)^{pq}[Y, X]_A$, for $X \in \Gamma(\bigwedge^{p+1} A)$, $Y \in \Gamma(\bigwedge^{q+1} A)$,
2. $[X, f]_A = a(X)f$ for $X \in \Gamma(A)$, $f \in C^\infty(M)$,
3. For $X \in \Gamma(\bigwedge^{p+1} A)$, $[X, \cdot]_A$ is a derivation of degree p of the exterior multiplication on $\Gamma(\bigwedge^* A)$.

One checks that this bracket satisfies the graded Jacobi identity with respect to the grading shifted down by one, and the resulting structure is a type of graded Poisson algebra called a *Gerstenhaber algebra*.

Dually, one gets a derivation d_A of degree 1 on the graded commutative algebra $\Gamma(\bigwedge^* A^*)$, defined by a formula identical to the Cartan formula for the de Rham differential:

$$\begin{aligned} d_A \omega(X_0, \dots, X_p) &= \sum_{i=0}^p (-1)^i a(X_i)(\omega(X_0, \dots, \hat{X}_i, \dots, X_p)) + \\ &+ \sum_{0 \leq i < j \leq p} (-1)^{i+j} \omega([X_i, X_j]_A, X_0, \dots, \hat{X}_i, \dots, \hat{X}_j, \dots, X_p), \end{aligned}$$

where $\omega \in \Gamma(\bigwedge^p A^*)$, and satisfying $d_A^2 = 0$. The space $\Gamma(\bigwedge^* A^*)$ thereby acquires the structure of a differential graded commutative algebra. d_A is uniquely determined by its action on $C^\infty(M)$ and $\Gamma(A^*)$:

$$\begin{aligned} d_A f(X) &= a(X)f \\ d_A \xi(X, Y) &= a(X)\xi(Y) - a(Y)\xi(X) - \xi([X, Y]_A) \end{aligned} \tag{2.3}$$

It is clear that, conversely, the Lie algebroid structure is completely determined by either d_A (all the structural identities are encoded in $d_A^2 = 0$), or the generalized Schouten bracket $[\cdot, \cdot]_A$.

Many notions of the usual calculus on manifolds carry over without change to Lie algebroids. In particular, for every $X \in \Gamma(A)$ there is a contraction (interior derivative) operator i_X acting on $\Gamma(\bigwedge^* A^*)$ by derivations of degree -1 , and the ‘‘Lie derivative’’ operator $L_X^A = [d_A, i_X]$ acting by derivations of degree 0 (here $[\cdot, \cdot]$ denotes the supercommutator).

These derivations satisfy the usual (super)commutation relations:

$$\begin{aligned} [d_A, d_A] &= 0, & [d_A, L_X^A] &= 0, & [d_A, i_X] &= L_X^A, \\ [L_X^A, L_Y^A] &= L_{[X, Y]_A}^A, & [i_X, i_Y] &= 0, & [L_X^A, i_Y] &= i_{[X, Y]_A} \end{aligned} \quad (2.4)$$

Now suppose that we are given a pair (A, A^*) of Lie algebroids over M which are in duality as vector bundles. Then the Lie algebroid structure of A induces a Schouten bracket on $\Gamma(\wedge^* A)$ and a differential d_A on $\Gamma(\wedge^* A^*)$; on the other hand, from A^* we get a Schouten bracket on $\Gamma(\wedge^* A^*)$ and a differential d_{A^*} on $\Gamma(\wedge^* A)$.

Definition 2.2.2. A pair (A, A^*) of Lie algebroids in duality is called a *Lie bialgebroid* if the induced differential d_A is a derivation of the Schouten bracket $[\cdot, \cdot]_{A^*}$ on $\Gamma(\wedge^* A^*)$.

Thus, Lie bialgebroids correspond to *differential Gerstenhaber algebras* [23]. The notion of a Lie bialgebroid is due to Mackenzie and Xu [35] who studied them and the corresponding global objects, Poisson groupoids (although the definition we quoted is an equivalent one from [23]). It can be shown that this notion is self-dual, i.e. if (A, A^*) is a Lie bialgebroid, so is (A^*, A) (Corollary 2.3.5 below).

Remark 2.2.3. Any Lie algebroid is a Lie bialgebroid with the zero anchor and bracket on the dual bundle.

Example 2.2.4. Let M be a manifold. Then its tangent bundle TM is a Lie algebroid whose bracket is the Jacobi-Lie bracket of vector fields, and the anchor is $\rho = \text{Id} : TM \rightarrow TM$. The corresponding extended bracket is the (original) Schouten bracket of multivector fields, while the differential is just the de Rham differential.

Example 2.2.5. Consider a (right) action of a Lie algebra \mathfrak{g} on a manifold M , i.e. a Lie algebra homomorphism $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$. This gives rise to a Lie algebroid structure on the trivial bundle $M \times \mathfrak{g} \rightarrow M$ whose anchor is given on constant sections by ρ and extended to all sections by linearity over $C^\infty(M)$, while the bracket of constant sections is just the bracket in \mathfrak{g} taken pointwise, which is then extended to all sections by the Leibniz rule. This Lie algebroid is called the *action Lie algebroid* associated to ρ . If ρ is a left action (a Lie algebra *antihomomorphism*), then we must take $-\rho$ as the anchor.

Example 2.2.6. If the base manifold M is a point, a Lie bialgebroid (A, A^*) over M is just a Lie bialgebra (we shall see later that Definition 2.2.2 is equivalent to Definition 2.1.1 in this case).

Example 2.2.7. Let M be a Poisson manifold with Poisson tensor π and the corresponding bundle map $\tilde{\pi} : T^*M \rightarrow TM$ given by $\langle \tilde{\pi}\alpha, \beta \rangle = \pi(\alpha, \beta)$. Let $A = TM$, the tangent bundle Lie algebroid, $A^* = T^*M$ with anchor $\tilde{\pi}$ and the bracket of 1-forms given by the *Koszul bracket*:

$$[\alpha, \beta]_{A^*} = \mathcal{L}_{\tilde{\pi}\alpha}\beta - \mathcal{L}_{\tilde{\pi}\beta}\alpha - d(\pi(\alpha, \beta)) \quad (2.5)$$

Then d_A is the usual de Rham differential of forms, $d_{A^*} = [\pi, \cdot]_A$, where $[\cdot, \cdot]_A$ is the Schouten bracket, and it is straightforward to verify that (A^*, A) is a Lie bialgebroid.

Detailed discussion and more examples of Lie bialgebroids and Gerstenhaber algebras from geometry and physics can be found in [23],[24] and [26].

2.3 Courant algebroids

Definition 2.3.1. Given a bilinear, skew-symmetric operation $[\cdot, \cdot]$ on a vector space V , its *Jacobiator* J is the trilinear operator on V :

$$J(e_1, e_2, e_3) = [[e_1, e_2], e_3] + [[e_2, e_3], e_1] + [[e_3, e_1], e_2],$$

$e_1, e_2, e_3 \in V$.

The Jacobiator is obviously skew-symmetric. Of course, in a Lie algebra $J \equiv 0$.

Definition 2.3.2. A *Courant algebroid* is a vector bundle $E \rightarrow M$ equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a skew-symmetric bracket $[\cdot, \cdot]$ on $\Gamma(E)$, and a bundle map $\rho : E \rightarrow TM$ such that the following properties are satisfied:

1. For any $e_1, e_2, e_3 \in \Gamma(E)$, $J(e_1, e_2, e_3) = \mathcal{D}T(e_1, e_2, e_3)$;
2. for any $e_1, e_2 \in \Gamma(E)$, $\rho[e_1, e_2] = [\rho e_1, \rho e_2]$;
3. for any $e_1, e_2 \in \Gamma(E)$ and $f \in C^\infty(M)$, $[e_1, f e_2] = f[e_1, e_2] + (\rho(e_1)f)e_2 - \frac{1}{2}\langle e_1, e_2 \rangle \mathcal{D}f$;
4. $\rho \circ \mathcal{D} = 0$, i.e., for any $f, g \in C^\infty(M)$, $\langle \mathcal{D}f, \mathcal{D}g \rangle = 0$;
5. for any $e, h_1, h_2 \in \Gamma(E)$, $\rho(e)\langle h_1, h_2 \rangle = \langle [e, h_1] + \frac{1}{2}\mathcal{D}\langle e, h_1 \rangle, h_2 \rangle + \langle h_1, [e, h_2] + \frac{1}{2}\mathcal{D}\langle e, h_2 \rangle \rangle$,

where $T(e_1, e_2, e_3)$ is the function on the base M defined by:

$$T(e_1, e_2, e_3) = \frac{1}{6} \langle [e_1, e_2], e_3 \rangle + c.p., \quad (2.6)$$

("c.p." denotes the cyclic permutations of the e_i 's) and $\mathcal{D} : C^\infty(M) \longrightarrow \Gamma(E)$ is the map defined by $\mathcal{D} = \rho^* d$, where E is identified with E^* by the bilinear form and d is the deRham differential. In other words,

$$\langle \mathcal{D}f, e \rangle = \rho(e)f. \quad (2.7)$$

Note. In our convention, the bilinear form $\langle \cdot, \cdot \rangle$ is two times the one in [31].

In a Courant algebroid E , a *Dirac structure*, or *Dirac subbundle*, is a subbundle L that is maximally isotropic under $\langle \cdot, \cdot \rangle$ and whose sections are closed under $[\cdot, \cdot]$. It is immediate from the definition that a Dirac subbundle is a Lie algebroid under the restrictions of the bracket and anchor.

Suppose now that both A and A^* are Lie algebroids over the base manifold M , with anchors a and a_* respectively. Let E denote their vector bundle direct sum: $E = A \oplus A^*$. On E , there exist two natural nondegenerate bilinear forms, one symmetric and another antisymmetric:

$$(X_1 + \xi_1, X_2 + \xi_2)_\pm = (\langle \xi_1, X_2 \rangle \pm \langle \xi_2, X_1 \rangle). \quad (2.8)$$

On $\Gamma(E)$, we introduce a bracket by

$$\begin{aligned} [e_1, e_2] &= ([X_1, X_2]_A + L_{\xi_1}^{A^*} X_2 - L_{\xi_2}^{A^*} X_1 - \frac{1}{2} d_{A^*}(e_1, e_2)_-) + \\ &+ ([\xi_1, \xi_2]_{A^*} + L_{X_1}^A \xi_2 - L_{X_2}^A \xi_1 + \frac{1}{2} d_A(e_1, e_2)_-), \end{aligned} \quad (2.9)$$

where $e_1 = X_1 + \xi_1$ and $e_2 = X_2 + \xi_2$.

Finally, we let $\rho : E \longrightarrow TM$ be the bundle map defined by $\rho = a + a_*$. That is,

$$\rho(X + \xi) = a(X) + a_*(\xi), \quad \forall X \in \Gamma(A) \text{ and } \xi \in \Gamma(A^*) \quad (2.10)$$

It is easy to see that in this case the operator \mathcal{D} as defined by Equation (2.7) is given by

$$\mathcal{D} = d_{A^*} + d_A$$

The following results, which we quote from [31], show that the notion of Courant algebroid permits us to generalize the double construction to Lie bialgebroids:

Theorem 2.3.3. *If (A, A^*) is a Lie bialgebroid, then $E = A \oplus A^*$ together with $([\cdot, \cdot], \rho, (\cdot, \cdot)_+)$ is a Courant algebroid.*

Theorem 2.3.4. *In a Courant algebroid $(E, \rho, [\cdot, \cdot], \langle \cdot, \cdot \rangle)$, suppose that L_1 and L_2 are Dirac subbundles transversal to each other, i.e., $E = L_1 \oplus L_2$. Then, (L_1, L_2) is a Lie bialgebroid, where L_2 is considered as the dual bundle of L_1 under the pairing $\langle \cdot, \cdot \rangle$.*

An immediate consequence of the theorems above is the following duality property of Lie bialgebroids, which was first proved in [35] and then by Kosmann-Schwarzbach [23] using a simpler method.

Corollary 2.3.5. *If (A, A^*) is a Lie bialgebroid, so is (A^*, A) .*

The theorems above are proved in [31] by rather laborious computations; in the next chapter we shall give a new, simple proof of Theorem 2.3.3 and Corollary 2.3.5.

Example 2.3.6. Given a manifold M , consider TM with its standard Lie algebroid structure and T^*M with zero anchor and bracket. Then (TM, T^*M) is a Lie bialgebroid, and the double bracket (2.9) reduces to

$$[X_1 + \xi_1, X_2 + \xi_2] = [X_1, X_2] + (L_{X_1}\xi_2 - L_{X_2}\xi_1 + d(\frac{1}{2}(\xi_1(X_2) - \xi_2(X_1)))).$$

This is the bracket originally introduced by Courant in [11]. The anchor ρ in this case is the projection to TM , and $\mathcal{D} = d$, the deRham differential.

Example 2.3.7. When M is a point, (A, A^*) is a Lie bialgebra and the bracket 2.9 on E becomes the famous Drinfeld double bracket.

2.4 Strongly homotopy Lie algebras and Courant algebroids

Let V be a graded vector space. Let $T(V)$ denote the tensor algebra of V in the category of graded vector spaces, and let $\bigwedge(V)$ denote its exterior algebra in the same category; i.e. $\bigwedge(V) = T(V) / \langle v \otimes w + (-1)^{\tilde{v}\tilde{w}} w \otimes v \rangle$, where \tilde{v} denotes the degree of v . $\bigwedge(V)$ has a natural Hopf algebra structure with the comultiplication $\Delta : \bigwedge(V) \rightarrow \bigwedge(V) \otimes \bigwedge(V)$ uniquely defined by the requirement that the elements of V be primitive (i.e.

$\Delta v = 1 \otimes v + v \otimes 1$ for $v \in V$) and that Δ be a homomorphism of algebras (see [28] for details).

Definition 2.4.1. A *strongly homotopy Lie algebra* (SHLA, L_∞ -algebra) is a graded vector space V together with a collection of linear maps $l_k : \bigwedge^k V \rightarrow V$ of degree $k - 2$, $k \geq 1$, satisfying the following relation for each $n \geq 1$ and for all homogeneous $x_1, \dots, x_n \in V$:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} \sum_{\sigma} (-1)^{\sigma} \epsilon(\sigma) l_j(l_i(x_{\sigma(1)}, \dots, x_{\sigma(i)}), x_{\sigma(i+1)}, \dots, x_{\sigma(n)}) = 0, \quad (2.11)$$

where σ runs over all $(i, n - i)$ -unshuffles (permutations satisfying $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(n)$) with $i \geq 1$, and $\epsilon(\sigma)$ is the *Koszul sign* (arising from the fundamental convention of supermathematics that a minus sign is introduced whenever two consecutive odd elements are permuted) .

For $n = 1$ this means simply that l_1 is a differential on V ; for $n = 2$, l_2 is a superbracket on V of which l_1 is a derivation (equivalently, $l_2 : \bigwedge^2(V) \rightarrow V$ is a chain map of complexes); $n = 3$ gives the Jacobi identity for l_2 satisfied up to chain homotopy given by l_3 , and higher l_k 's can be interpreted as higher homotopies. The algebraic theory of L_∞ -algebras is studied in [19] and [28] .

We shall write the equation (2.11) in the more succinct equivalent form:

$$\sum_{i+j=n+1} (-1)^{i(j-1)} l_j l_i = 0, \quad (2.12)$$

where we have extended each l_i to all of $\bigwedge(V)$ as a coderivation of the coalgebra structure on $\bigwedge(V)$. This accounts for the permutations and signs in (2.11).

We are interested in L_∞ -algebras for the following reason: it is shown in [7] that, given a resolution (X_*, d) of a vector space H (graded or not), any Lie algebra structure on H can be lifted to an L_∞ -algebra structure on the total resolution space X with $l_1 = d$. The starting point of this construction is the observation that Lie brackets on H correspond to bilinear skew-symmetric brackets $[\cdot, \cdot]$ on X_0 for which the boundaries form an ideal and the Jacobi identity is satisfied up to a boundary. This correspondence is in no way unique or canonical, as it requires a choice of a homotopy inverse to the quasi-isomorphism $(X_*, d) \rightarrow (H, 0)$. But it is this bracket $[\cdot, \cdot]$ on X_0 that provides the starting point for constructing the SHLA structure on X , hence, if it is given, no choice is required at this

stage, and we need never mention H . We shall presently see that with Courant algebroids we are in precisely this situation.

Let E be a Courant algebroid over a manifold M . We know from the definition that the Courant bracket on $\Gamma(E)$ satisfies Jacobi up to a \mathcal{D} -exact term. It turns out that, moreover, $\text{Im}(\mathcal{D})$ is an ideal in $\Gamma(E)$ with respect to the bracket. More precisely, the following identity holds:

Lemma 2.4.2. *For any $e \in \Gamma(E)$, $f \in C^\infty(M)$ one has*

$$[e, \mathcal{D}f] = \frac{1}{2} \mathcal{D}\langle e, \mathcal{D}f \rangle$$

Proof. Use axiom 5 in the definition of Courant algebroid with $e = \mathcal{D}f$ and arbitrary h_1 and h_2 , and then cyclically permute e , h_1 and h_2 :

$$\begin{aligned} \rho(\mathcal{D}f)\langle h_1, h_2 \rangle &= \langle [\mathcal{D}f, h_1] + \frac{1}{2} \mathcal{D}\langle \mathcal{D}f, h_1 \rangle, h_2 \rangle + \langle h_1, [\mathcal{D}f, h_2] + \frac{1}{2} \mathcal{D}\langle \mathcal{D}f, h_2 \rangle \rangle \\ \rho(h_1)\langle h_2, \mathcal{D}f \rangle &= \langle [h_1, h_2] + \frac{1}{2} \mathcal{D}\langle h_1, h_2 \rangle, \mathcal{D}f \rangle + \langle h_2, [h_1, \mathcal{D}f] + \frac{1}{2} \mathcal{D}\langle h_1, \mathcal{D}f \rangle \rangle \\ \rho(h_2)\langle \mathcal{D}f, h_1 \rangle &= \langle [h_2, \mathcal{D}f] + \frac{1}{2} \mathcal{D}\langle h_2, \mathcal{D}f \rangle, h_1 \rangle + \langle \mathcal{D}f, [h_2, h_1] + \frac{1}{2} \mathcal{D}\langle h_2, h_1 \rangle \rangle. \end{aligned}$$

Now add the first two identities and subtract the third. Using Courant algebroid axioms 2, 4 and the definition of \mathcal{D} , we get:

$$\rho([h_1, h_2])f = \langle \mathcal{D}f, 2[h_1, h_2] \rangle + \langle h_1, 2[\mathcal{D}f, h_2] \rangle + \langle h_2, \mathcal{D}\langle \mathcal{D}f, h_1 \rangle \rangle.$$

Using the definition of \mathcal{D} again, we can rewrite this as:

$$\begin{aligned} 0 &= \rho([h_1, h_2])f + \langle h_1, 2[\mathcal{D}f, h_2] \rangle + \rho(h_2)\langle h_1, \mathcal{D}f \rangle = \\ &= \rho([h_1, h_2])f + \langle h_1, 2[\mathcal{D}f, h_2] \rangle + \rho(h_2)(\rho(h_1)f) = \\ &= \rho(h_1)(\rho(h_2)f) + \langle h_1, 2[\mathcal{D}f, h_2] \rangle = \\ &= \langle h_1, \mathcal{D}(\rho(h_2)f) + 2[\mathcal{D}f, h_2] \rangle = \\ &= \langle h_1, 2(\frac{1}{2} \mathcal{D}\langle h_2, \mathcal{D}f \rangle - [h_2, \mathcal{D}f]) \rangle. \end{aligned}$$

The statement follows from the nondegeneracy of $\langle \cdot, \cdot \rangle$. □

It will follow that we can extend the Courant bracket to an L_∞ -structure on the total space of the following resolution of $H = \text{coker } \mathcal{D}$:

$$\cdots \longrightarrow 0 \longrightarrow X_2 \xrightarrow{d_2} X_1 \xrightarrow{d_1} X_0 \longrightarrow H \longrightarrow 0, \quad (2.13)$$

where $X_0 = \Gamma(E)$, $X_1 = C^\infty(M)$, $X_2 = \ker \mathcal{D}$, $d_1 = \mathcal{D}$ and d_2 is the inclusion $\iota : \ker \mathcal{D} \hookrightarrow C^\infty(M)$. Remarkably, it turns out that, owing to the properties of Courant algebroids, the choices in the extension procedure can be made in a natural and simple way.

Let us fix some notation: we will denote elements of X_0 by e , elements of X_1 by f or g , and elements of X_2 by c .

Theorem 2.4.3. *A Courant algebroid structure on a vector bundle $E \longrightarrow M$ gives rise naturally to a SHLA structure on the total space X of (2.13) with $l_1 = d$ and the higher structure maps given by the following explicit formulas:*

$$\begin{aligned}
l_2(e_1 \wedge e_2) &= [e_1, e_2] && \text{in degree } 0 \\
l_2(e \wedge f) &= \frac{1}{2}\langle e, \mathcal{D}f \rangle && \text{in degree } 1 \\
l_2 &= 0 && \text{in degree } > 1 \\
l_3(e_1 \wedge e_2 \wedge e_3) &= -T(e_1, e_2, e_3) && \text{in degree } 0 \\
l_3 &= 0 && \text{in degree } > 0 \\
l_n &= 0 && \text{for } n > 3
\end{aligned}$$

Proof. Starting with the Courant bracket on X_0 , we shall, following [7], extend it to an l_2 on all of X satisfying (2.12) for $n = 2$. The extension will proceed, essentially, by induction on the degree of the argument: for each degree l_2 will be a primitive of a certain cycle depending on the values of l_2 on elements of lower degree. Higher l_k 's will be introduced and extended in a similar fashion, as primitives of cycles (using the acyclicity of (2.13)). The main work will consist in calculating these cycles, in particular, showing that most of them vanish; these computations are mostly relegated to the technical lemmas of the next section. *Step 1: $n = 2$.* In degree 0, we are given $l_2(e_1 \wedge e_2) = [e_1, e_2]$. Consider now an element $e \wedge f$ of degree 1. Then $l_2 l_1(e \wedge f) \in X_0$ is defined and is, in fact, a boundary by Lemma 2.4.2:

$$l_2 l_1(e \wedge f) = l_2(l_1 e \wedge f + e \wedge l_1 f) = [e, \mathcal{D}f] = \frac{1}{2}\mathcal{D}\langle e, \mathcal{D}f \rangle,$$

so we set $l_2(e \wedge f) = \frac{1}{2}\langle e, \mathcal{D}f \rangle$ so that the SHLA identity (2.12) for $n = 2$,

$$l_1 l_2 - l_2 l_1 = 0, \tag{2.14}$$

holds in degree 1. Now, $\bigwedge^2(X)_2$ is spanned by elements of the form $f \wedge g$ or $c \wedge e$. As above, $l_2 l_1$ is defined on elements of degree 2, and is, in fact, a cycle (cf. [7]). We have

$$l_2 l_1(f \wedge g) = l_2(l_1 f \wedge g - f \wedge l_1 g) = l_2(\mathcal{D}f \wedge g - f \wedge \mathcal{D}g) = \frac{1}{2}(\langle \mathcal{D}f, \mathcal{D}g \rangle + \langle \mathcal{D}g, \mathcal{D}f \rangle) = 0$$

by Courant algebroid axiom 4, whereas

$$l_2 l_1(c \wedge e) = l_2(l_1 c \wedge e + c \wedge l_1 e) = l_2(\iota c \wedge e) = -\frac{1}{2}\langle e, \mathcal{D}\iota c \rangle = 0,$$

so we set $l_2(f \wedge g) = l_2(c \wedge e) = 0$. Now observe that, since $l_2 = 0$ in degree 2, we can define l_2 to be zero on elements of degree higher than 2 as well and still have (2.14). We have thus defined an l_2 that satisfies (2.14) by construction. *Step 2: $n = 3$.* In degree 0, by Courant algebroid axiom 1 we have

$$l_2 l_2(e_1 \wedge e_2 \wedge e_3) = J(e_1, e_2, e_3) = \mathcal{D}T(e_1, e_2, e_3),$$

where J is the Jacobiator. So we set $l_3(e_1 \wedge e_2 \wedge e_3) = -T(e_1, e_2, e_3)$, so that the homotopy Jacobi identity identity (2.12) for $n = 3$,

$$l_1 l_3 + l_2 l_2 + l_3 l_1 = 0, \tag{2.15}$$

holds on $\bigwedge^3(X)_0$ (as $l_1(X_0) = 0$). Consider now an element $e_1 \wedge e_2 \wedge f \in \bigwedge^3(X)_1$. The expression $(l_2 l_2 + l_3 l_1)(e_1 \wedge e_2 \wedge f)$ is defined and is a cycle in X_1 (cf. [7]), hence we can define $l_3(e_1 \wedge e_2 \wedge f)$ to be some primitive of this cycle, so that (2.15) holds. But in our particular situation we in fact have (see the next section for a proof):

Lemma 2.4.4. $(l_2 l_2 + l_3 l_1)(e_1 \wedge e_2 \wedge f) = 0 \ \forall e_1, e_2, f$.

Therefore, we can define $l_3(e_1 \wedge e_2 \wedge f) = 0$. Now observe that on elements of degree > 1 l_3 has to be 0 because $\deg(l_3) = 1$, whereas $X_k = 0$ for $k > 2$. We now have l_3 defined on all of $\bigwedge^3(X)$ and satisfying (2.15) by construction. *Step 3: $n = 4$ and higher.* Proceeding in a similar fashion, we look at the expression

$(l_3 l_2 - l_2 l_3)(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$ (always a cycle in X_1) and define $l_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4)$ to be its primitive in X_2 , so as to satisfy (2.12). However, it turns out that (see the next section for a proof)

Lemma 2.4.5. $(l_3 l_2 - l_2 l_3)(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 0 \ \forall e_1, e_2, e_3, e_4$.

Hence we can set $l_4(e_1 \wedge e_2 \wedge e_3 \wedge e_4) = 0$ and observe that l_4 has to vanish on elements of degree > 0 as $\deg(l_4) = 2$, while $X_k = 0$ for $k > 2$. By similar degree counting, all l_n , $n > 4$, have to vanish identically. This finishes the proof modulo Lemmas 2.4.4 and 2.4.5. \square

Remark 2.4.6. If the base M is a point, a Courant algebroid reduces to a Lie algebra \mathfrak{p} with an invariant inner product; however, even though the differential \mathcal{D} is trivial in this case and all the anomalies vanish, the homotopy Lie algebra we get is not “just a Lie algebra”: in addition to the Lie algebra bracket there is also a trilinear operation T , the structure tensor of the Lie algebra:

$$T(X, Y, Z) = \frac{1}{2}\langle [X, Y], Z \rangle$$

for $X, Y, Z \in \mathfrak{p}$.

2.5 Proofs of technical lemmas

Let $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ be a Courant algebroid over M . Given $e \in \Gamma(E)$, $f \in C^\infty(M)$, we will denote $\rho(e)f$ simply by ef , for short. Let us first prove two auxiliary lemmas.

Lemma 2.5.1. *The identity*

$$T(e_1, e_2, \mathcal{D}f) = \frac{1}{4}[e_1, e_2]f$$

holds in any Courant algebroid.

Proof. Using Courant algebroid axiom 2 and Lemma 2.4.2, we have

$$\begin{aligned} T(e_1, e_2, \mathcal{D}f) &= \frac{1}{6}(\langle [e_1, e_2], \mathcal{D}f \rangle + \langle [\mathcal{D}f, e_1], e_2 \rangle + \langle [e_2, \mathcal{D}f], e_1 \rangle) = \\ &= \frac{1}{6}(\langle [e_1, e_2], \mathcal{D}f \rangle - \frac{1}{2}\langle \mathcal{D}\langle e_1, \mathcal{D}f \rangle, e_2 \rangle + \frac{1}{2}\langle \mathcal{D}\langle e_2, \mathcal{D}f \rangle, e_1 \rangle) = \\ &= \frac{1}{6}([e_1, e_2]f - \frac{1}{2}e_2(e_1f) + \frac{1}{2}e_1(e_2f)) = \\ &= \frac{1}{6}([e_1, e_2]f + \frac{1}{2}[e_1, e_2]f) = \frac{1}{4}[e_1, e_2]f. \end{aligned}$$

□

Lemma 2.5.2. *Given $e_1, e_2, e_3, e_4 \in \Gamma(E)$, let*

$$\begin{aligned} \mathbf{J} &= \langle J(e_1, e_2, e_3), e_4 \rangle - \langle J(e_1, e_2, e_4), e_3 \rangle + \langle J(e_1, e_3, e_4), e_2 \rangle - \langle J(e_2, e_3, e_4), e_1 \rangle \\ \mathbf{K} &= \langle [e_1, e_2], [e_3, e_4] \rangle - \langle [e_1, e_3], [e_2, e_4] \rangle + \langle [e_1, e_4], [e_2, e_3] \rangle, \end{aligned}$$

where J is the Jacobiator (cf. Def 2.3.1). Then $\mathbf{K} + 2\mathbf{J} = 0$.

Proof. Using Courant algebroid axioms 1 and 5, we can rewrite \mathbf{J} as follows:

$$\begin{aligned}\langle J(e_1, e_2, e_3), e_4 \rangle &= \langle \mathcal{D}T(e_1, e_2, e_3), e_4 \rangle = e_4 T(e_1, e_2, e_3) = \frac{1}{6} e_4 (\langle [e_1, e_2], e_3 \rangle + c.p.) = \\ &= \frac{1}{6} (\langle [e_4, [e_1, e_2]] + \frac{1}{2} \mathcal{D}\langle e_4, [e_1, e_2] \rangle, e_3 \rangle + \langle [e_1, e_2], [e_4, e_3] + \frac{1}{2} \mathcal{D}\langle e_4, e_3 \rangle \rangle) + c.p.\end{aligned}$$

Expressing the other summands of \mathbf{J} in this form and collecting like terms in the parentheses, we find that the terms of the form $\langle [e_i, e_j], \mathcal{D}\langle e_k, e_l \rangle \rangle$ cancel out, terms of the form $\langle [e_i, e_j], [e_k, e_l] \rangle$ add up to $-4\mathbf{K}$, those of the form $\langle [e_i, [e_j, e_k]], e_l \rangle$ add up to \mathbf{J} , and finally, terms of the form $\langle \mathcal{D}\langle e_i, [e_j, e_k] \rangle, e_l \rangle$ add up to $-3\mathbf{J}$ after we use Courant algebroid axiom 1. Thus,

$$\mathbf{J} = \frac{1}{6}(\mathbf{J} - 3\mathbf{J} - 4\mathbf{K}),$$

and the statement of the lemma follows immediately. \square

Proof of Lemma 2.4.4. In the notation of the previous section, we have, using Lemma 2.5.1 and Courant algebroid axiom 2:

$$\begin{aligned}(l_2 l_2 + l_3 l_1)(e_1 \wedge e_2 \wedge f) &= \\ &= l_2(l_2(e_1 \wedge e_2) \wedge f + l_2(e_2 \wedge f) \wedge e_1 + l_2(f \wedge e_1) \wedge e_2) + \\ &+ l_3(l_1 e_1 \wedge e_2 \wedge f + e_1 \wedge l_1 e_2 \wedge f + e_1 \wedge e_2 \wedge l_1 f) = \\ &= l_2([e_1, e_2] \wedge f + \frac{1}{2} \langle e_2, \mathcal{D}f \rangle \wedge e_1 - \frac{1}{2} \langle \mathcal{D}f, e_1 \rangle \wedge e_2) + l_3(e_1 \wedge e_2 \wedge \mathcal{D}f) = \\ &= \frac{1}{2} \langle [e_1, e_2], \mathcal{D}f \rangle - \frac{1}{4} \langle e_1, \mathcal{D}\langle e_2, \mathcal{D}f \rangle \rangle + \frac{1}{4} \langle e_2, \mathcal{D}\langle e_1, \mathcal{D}f \rangle \rangle - T(e_1, e_2, \mathcal{D}f) = \\ &= \frac{1}{2} [e_1, e_2] f - \frac{1}{4} e_1 (e_2 f) + \frac{1}{4} e_2 (e_1 f) - \frac{1}{4} [e_1, e_2] f = 0\end{aligned}$$

\square

Proof of Lemma 2.4.5. In the notation of the previous section we have

$$\begin{aligned}l_2 l_3(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= l_2(l_3(e_1 \wedge e_2 \wedge e_3) \wedge e_4 \pm (3, 1) - unshuffles) = \\ &= -l_2(T(e_1, e_2, e_3) \wedge e_4 \pm (3, 1) - unshuffles) = \\ &= \frac{1}{2} \langle \mathcal{D}T(e_1, e_2, e_3), e_4 \rangle \pm (3, 1) - unshuffles = \\ &= \frac{1}{2} \langle J(e_1, e_2, e_3), e_4 \rangle \pm (3, 1) - unshuffles = \frac{1}{2} \mathbf{J}.\end{aligned}$$

On the other hand,

$$\begin{aligned}
l_3 l_2(e_1 \wedge e_2 \wedge e_3 \wedge e_4) &= l_3(l_2(e_1 \wedge e_2) \wedge e_3 \wedge e_4) \pm (2, 2) - \text{unshuffles} = \\
&= -T([e_1, e_2], e_3, e_4) \mp (2, 2) - \text{unshuffles} = \\
&= -\frac{1}{6}(\langle [e_1, e_2], e_3, e_4 \rangle + \langle [e_3, e_4], [e_1, e_2] \rangle + \langle [e_4, [e_1, e_2]], e_3 \rangle) \\
&\pm \dots = -\frac{1}{6}(\mathbf{J} + 2\mathbf{K}),
\end{aligned}$$

after collecting like terms. An application of Lemma 2.5.2 immediately yields $l_2 l_3 = l_3 l_2$. \square

2.6 Alternative definition of Courant algebroid

Definition 2.6.1. A *Courant algebroid* is a vector bundle $E \rightarrow M$ together with a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$ on the bundle, a bilinear operation \circ on $\Gamma(E)$, and a bundle map $\rho : E \rightarrow TM$ satisfying the following properties:

1. $e_1 \circ (e_2 \circ e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3) \quad \forall e_1, e_2, e_3 \in \Gamma(E)$;
2. $\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)] \quad \forall e_1, e_2 \in \Gamma(E)$;
3. $e_1 \circ f e_2 = f(e_1 \circ e_2) + (\rho(e_1) \cdot f) e_2 \quad \forall e_1, e_2 \in \Gamma(E), f \in C^\infty(M)$;
4. $e \circ e = \frac{1}{2} \mathcal{D} \langle e, e \rangle \quad \forall e \in \Gamma(E)$;
5. $\rho(e) \cdot \langle h_1, h_2 \rangle = \langle e \circ h_1, h_2 \rangle + \langle h_1, e \circ h_2 \rangle \quad \forall e, h_1, h_2 \in \Gamma(E)$,

where $\mathcal{D} : C^\infty(M) \rightarrow \Gamma(E)$ is given by (2.7).

Notice that all the anomalies of Definition 2.3.2 are absent in this one, but the skew-symmetric bracket $[\cdot, \cdot]$ is replaced by a not necessarily skew-symmetric operation \circ .¹ Property 1 above is to be interpreted as the “Jacobi identity” for \circ in the sense that $e \circ \cdot$ is a derivation of \circ for any $e \in \Gamma(E)$;² if \circ is skew-symmetric, this is equivalent to the usual

¹ This was first proposed in [31], but the properties of this operation were then an open question; this definition of a Courant algebroid was also used in a note of P. Ševera [43], without a proof of its equivalence to the original one.

² It looks more like a Leibniz rule; in fact, a vector space with a bilinear operation satisfying this property was called a Leibniz algebra by Loday [32], and a Loday algebra by Kosmann-Schwarzbach [25].

Jacobi identity. On the other hand, Property 4 is equivalent to saying, by a polarization identity, that \circ is skew-symmetric “up to a coboundary”, i.e. the symmetric part of it is \mathcal{D} of something. More precisely, we have

$$e_1 \circ e_2 = [e_1, e_2] + \frac{1}{2} \mathcal{D} \langle e_1, e_2 \rangle \quad (2.16)$$

for all $e_1, e_2 \in \Gamma(E)$, where

$$[e_1, e_2] = \frac{1}{2} (e_1 \circ e_2 - e_2 \circ e_1) \quad (2.17)$$

is the skew-symmetrization of \circ . We shall now prove that the new Definition 2.6.1 is equivalent to the old Definition 2.3.2. We need a couple of lemmas first. The first one is the non-skew-symmetric version of Lemma 2.4.2.

Lemma 2.6.2. *If $(E, \langle \cdot, \cdot \rangle, \circ, \rho)$ satisfies properties 2-5 of Definition 2.6.1, then $\forall e \in \Gamma(E)$, $f \in C^\infty(M)$ one has*

$$\begin{aligned} e \circ \mathcal{D}f &= \mathcal{D} \langle e, \mathcal{D}f \rangle \\ \mathcal{D}f \circ e &= 0 \end{aligned}$$

Proof. Let $h \in \Gamma(E)$ be arbitrary. Then, using Properties 2 and 5 we have

$$\begin{aligned} \rho(e)(\rho(h)f) &= \rho(e) \langle \mathcal{D}f, h \rangle = \langle e \circ \mathcal{D}f, h \rangle + \langle \mathcal{D}f, e \circ h \rangle = \\ &= \langle e \circ \mathcal{D}f, h \rangle + \rho(e \circ h)f = \\ &= \langle e \circ \mathcal{D}f, h \rangle + \rho(e)(\rho(h)f) - \rho(h)(\rho(e)f) \end{aligned}$$

Hence,

$$\langle e \circ \mathcal{D}f, h \rangle = \rho(h)(\rho(e)f) = \langle h, \mathcal{D} \langle e, \mathcal{D}f \rangle \rangle$$

The first statement follows by the nondegeneracy of $\langle \cdot, \cdot \rangle$. On the other hand, by (2.16),

$$\mathcal{D}f \circ e = \mathcal{D}f \circ e + e \circ \mathcal{D}f - e \circ \mathcal{D}f = \mathcal{D} \langle e, \mathcal{D}f \rangle - \mathcal{D} \langle e, \mathcal{D}f \rangle = 0$$

□

Remark 2.6.3. Observe that the statement of Lemma 2.6.2 is equivalent to the statement of Lemma 2.4.2 for the skew-symmetrization (2.17), in view of (2.16).

Lemma 2.6.4. *If $(E, \langle \cdot, \cdot \rangle, \circ, \rho)$ satisfies properties 2-5 of definition 2.6.1, then the expression*

$$K(e_1, e_2, e_3) = (e_1 \circ e_2) \circ e_3 + e_2 \circ (e_1 \circ e_3) - e_1 \circ (e_2 \circ e_3)$$

is completely skew-symmetric in e_1, e_2, e_3 .

Proof. We have to show that K vanishes if any two of the entries coincide. But

$$K(e_1, e_1, e_3) = (e_1 \circ e_1) \circ e_3 + e_1 \circ (e_1 \circ e_3) - e_1 \circ (e_1 \circ e_3) = \frac{1}{2} \mathcal{D}\langle e_1, e_1 \rangle \circ e_3 = 0$$

by property 4 and Lemma 2.6.2. On the other hand,

$$\begin{aligned} K(e_1, e_2, e_2) &= (e_1 \circ e_2) \circ e_2 + e_2 \circ (e_1 \circ e_2) - e_1 \circ (e_2 \circ e_2) = \\ &= \mathcal{D}(\langle e_1 \circ e_2, e_2 \rangle - \langle e_1, e_2 \circ e_2 \rangle) = \\ &= \mathcal{D}(\langle e_1 \circ e_2, e_2 \rangle + \langle e_2 \circ e_1, e_2 \rangle - \langle e_2, \mathcal{D}\langle e_1, e_2 \rangle \rangle) = \\ &= \mathcal{D}(\langle \mathcal{D}\langle e_1, e_2 \rangle, e_2 \rangle - \langle e_2, \mathcal{D}\langle e_1, e_2 \rangle \rangle) = 0, \end{aligned}$$

where we have used properties 4 and 5 and Lemma 2.6.2. And finally,

$$\begin{aligned} K(e_1, e_2, e_1) &= (e_1 \circ e_2) \circ e_1 + e_2 \circ (e_1 \circ e_1) - e_1 \circ (e_2 \circ e_1) = \\ &= (e_1 \circ e_2) \circ e_1 + (e_2 \circ e_1) \circ e_1 + e_2 \circ (e_1 \circ e_1) - (e_2 \circ e_1) \circ e_1 - e_1 \circ (e_2 \circ e_1) = \\ &= \mathcal{D}\langle e_1, e_2 \rangle \circ e_1 - \mathcal{D}\langle e_2 \circ e_1, e_1 \rangle + \mathcal{D}\langle e_2, e_1 \circ e_1 \rangle = \\ &= -\mathcal{D}(\langle e_2 \circ e_1, e_1 \rangle - \langle e_2, e_1 \circ e_1 \rangle) = 0, \end{aligned}$$

just as in the previous calculation; we have used again properties 4 and 5 and Lemma 2.6.2. \square

We are now ready to prove the equivalence of the two definitions of Courant algebroid.

Proposition 2.6.5. *Let $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ be a Courant algebroid in the sense of Definition 2.3.2. Let the operation \circ be given by (2.16). Then $(E, \langle \cdot, \cdot \rangle, \circ, \rho)$ is a Courant algebroid in the sense of Definition 2.6.1.*

Conversely, let $(E, \langle \cdot, \cdot \rangle, \circ, \rho)$ be a Courant algebroid in the sense of Definition 2.6.1. Let $[\cdot, \cdot]$ be the skew-symmetrization of \circ , as in (2.17). Then $(E, \langle \cdot, \cdot \rangle, [\cdot, \cdot], \rho)$ is a Courant algebroid in the sense of Definition 2.3.2.

Proof. It is not hard to show the equivalence of all of the properties, except for the Jacobi identity which will take a bit more work. So we shall first show the equivalence of properties

2-5 for both definitions, and then prove the equivalence of the two versions of Jacobi using Lemmas 2.6.2 and 2.6.4.

Now, in view of (2.16) it is obvious that Property 5 is equivalent for the two definitions. Property 2 in the new definition implies immediately that $\rho(e \circ e) = 0$ for all $e \in \Gamma(E)$, hence

$$[\rho(e_1), \rho(e_2)] = \rho(e_1 \circ e_2) = \rho([e_1, e_2]),$$

and we have the old Property 2. Moreover, by the new Property 4,

$$0 = \rho(e_1 \circ e_2 + e_2 \circ e_1) = \frac{1}{2} \mathcal{D} \langle e_1, e_2 \rangle \quad \forall e_1, e_2 \in \Gamma(E),$$

hence we have the old Property 4 ($\rho \circ \mathcal{D} = 0$) by the nondegeneracy of $\langle \cdot, \cdot \rangle$. Conversely, if we start with the old definition, the new Property 4 is immediate by (2.16), while the old Properties 2 and 4 combine to give

$$\rho(e_1 \circ e_2) = \rho([e_1, e_2] + \frac{1}{2} \mathcal{D} \langle e_1, e_2 \rangle) = \rho([e_1, e_2]) = [\rho(e_1), \rho(e_2)],$$

the new Property 2.

As for the Leibniz rule, one has

$$\begin{aligned} e_1 \circ f e_2 &= [e_1, f e_2] + \frac{1}{2} \mathcal{D} \langle e_1, f e_2 \rangle = \\ &= [e_1, f e_2] + \frac{1}{2} f \mathcal{D} \langle e_1, e_2 \rangle + \frac{1}{2} \langle e_1, e_2 \rangle \mathcal{D} f \end{aligned}$$

for all $e_1, e_2 \in \Gamma(E)$, $f \in C^\infty(M)$, hence it follows immediately that the new and old Properties 3 are equivalent.

Now for the Jacobi identity. In view of (2.16), it is clear that one has

$$K(e_1, e_2, e_3) = J(e_1, e_2, e_3) + R(e_1, e_2, e_3),$$

where K is as in Lemma 2.6.4, J is the Jacobiator (Def. 2.3.1), and

$$\begin{aligned} R(e_1, e_2, e_3) &= \frac{1}{2} ([\mathcal{D} \langle e_1, e_2 \rangle, e_3] + [e_2, \mathcal{D} \langle e_1, e_3 \rangle] - [e_1, \mathcal{D} \langle e_2, e_3 \rangle]) + \\ &+ \frac{1}{2} \mathcal{D} (\langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle - \langle e_1, e_2 \circ e_3 \rangle). \end{aligned}$$

To show the equivalence of the old and new Properties 1, we only need to show that $R(e_1, e_2, e_3) = -\mathcal{D}T(e_1, e_2, e_3)$, where T is as in (2.6). But, by Lemma 2.4.2 and Remark 2.6.3,

$$\begin{aligned} &\frac{1}{2} ([\mathcal{D} \langle e_1, e_2 \rangle, e_3] + [e_2, \mathcal{D} \langle e_1, e_3 \rangle] - [e_1, \mathcal{D} \langle e_2, e_3 \rangle]) = \\ &= -\frac{1}{4} \mathcal{D} (\langle \mathcal{D} \langle e_1, e_2 \rangle, e_3 \rangle - \langle e_2, \mathcal{D} \langle e_1, e_3 \rangle \rangle + \langle e_1, \mathcal{D} \langle e_2, e_3 \rangle \rangle), \end{aligned}$$

whereas

$$\begin{aligned} & \frac{1}{2}\mathcal{D}(\langle e_1 \circ e_2, e_3 \rangle + \langle e_2, e_1 \circ e_3 \rangle - \langle e_1, e_2 \circ e_3 \rangle) = \\ & = \frac{1}{2}\mathcal{D}(\langle [e_1, e_2], e_3 \rangle + \langle e_2, [e_1, e_3] \rangle - \langle e_1, [e_2, e_3] \rangle) + \\ & + \frac{1}{4}\mathcal{D}(\langle \mathcal{D}\langle e_1, e_2 \rangle, e_3 \rangle + \langle e_2, \mathcal{D}\langle e_1, e_3 \rangle \rangle - \langle e_1, \mathcal{D}\langle e_2, e_3 \rangle \rangle) \end{aligned}$$

by (2.16). Therefore,

$$\begin{aligned} R(e_1, e_2, e_3) &= \frac{1}{2}\mathcal{D}(\langle [e_1, e_2], e_3 \rangle - \langle [e_3, e_1], e_2 \rangle - \langle [e_2, e_3], e_1 \rangle) + \\ &+ \frac{1}{2}\mathcal{D}(\langle e_2, \mathcal{D}\langle e_1, e_3 \rangle \rangle - \langle e_1, \mathcal{D}\langle e_2, e_3 \rangle \rangle) \end{aligned} \quad (2.18)$$

However, since both J and K are completely antisymmetric (Lemma 2.6.4), so is R ; therefore, R is equal to its skew-symmetrization. But it is obvious that the skew-symmetrization of the first term on the right hand side of (2.18) is $-\mathcal{D}T(e_1, e_2, e_3)$, whereas the skew-symmetrization of the second term is easily seen to be zero. Hence

$R(e_1, e_2, e_3) = -\mathcal{D}T(e_1, e_2, e_3)$, and we are done. \square

Remark 2.6.6. Notice that the notion of a Dirac subbundle remains unchanged when we switch to the new definition of a Courant algebroid, thanks to (2.16), and that the restrictions of the two operations to any Dirac subbundle are identical.

Example 2.6.7. Let (A, A^*) be a pair of Lie algebroids in duality, with anchors a and a_* , respectively. Then on $E = A \oplus A^*$ we define

$$\begin{aligned} \langle X + \xi, Y + \eta \rangle &= \xi(Y) + \eta(X) \\ (X + \xi) \circ (Y + \eta) &= ([X, Y]_A + L_\xi^{A^*} Y - i_\eta d_{A^*} X) + \\ &+ ([\xi, \eta]_{A^*} + L_X^A \eta - i_Y d_A \xi) \\ \rho(X + \xi) &= a(X) + a_*(\xi) \end{aligned} \quad (2.19)$$

If (A, A^*) is a Lie bialgebroid, then by Theorem 2.3.3 and Proposition 2.6.5 $(E, \langle \cdot, \cdot \rangle, \circ, \rho)$ is a Courant algebroid in the sense of Definition 2.6.1.

Example 2.6.8. As a special case of Example 2.6.7, consider TM with its standard Lie algebroid structure and T^*M with the zero anchor and bracket. Then on sections of $TM \oplus T^*M$ the operation \circ reduces to

$$(X + \xi) \circ (Y + \eta) = [X, Y] + L_X \eta - i_Y d\xi \quad (2.20)$$

whose skew-symmetrization was originally considered by Courant in his study of Dirac manifolds [11]. This Courant algebroid was also considered by P. Ševera [43] as the natural geometric framework for two-dimensional variational problems.

Chapter 3

The double of a Lie bialgebroid as a homological vector field on an even symplectic supermanifold

We shall now present an alternative construction of the double of a Lie bialgebroid. It is based on an interpretation of a Lie algebroid as an odd self-commuting vector field on a supermanifold, which we then lift as a hamiltonian on its cotangent bundle. Adding the two hamiltonians coming from the dual Lie algebroids, we get a third one which Poisson-commutes with itself if and only if the compatibility condition of a Lie bialgebroid is satisfied; the corresponding hamiltonian vector field is interpreted as the Drinfeld double. As an application, we show that the Weil differential and the classical BRST differential arise in this way. The Courant algebroid of Example 2.6.7 is recovered via the derived bracket construction; this allows us to give a simple proof of the doubling theorem of Liu, Weinstein and Xu [31]. Finally, we consider quasi-Lie bialgebroids which one gets by adding cubic terms to the hamiltonian and show that exact Courant algebroids, recently classified by Ševera [43], arise in this way.

The starting point for us is a picture of Lie bialgebras due to Lecomte, Roger and Kosmann-Schwarzbach.

3.1 An alternative picture of Lie bialgebras

There is an elegant way to express the structure relations of a Lie bialgebra by embedding it into a larger space endowed with a canonical Poisson superalgebra structure [29] [22]. By viewing this construction from an appropriate angle we shall be able to generalize it to Lie bialgebroids, obtain a new notion of the Drinfeld double and recover the old one.

Consider a Lie bialgebra $(\mathfrak{g}, \mu, \gamma)$ (see Definition 2.1.1); view the bracket μ and the cobracket γ as elements $\mu \in \wedge^2 \mathfrak{g}^* \otimes \mathfrak{g}$ and $\gamma \in \mathfrak{g}^* \otimes \wedge^2 \mathfrak{g}$.

The basic idea is to embed μ and γ into the full exterior algebra $\wedge(\mathfrak{g} \oplus \mathfrak{g}^*) = (\wedge \mathfrak{g}) \otimes (\wedge \mathfrak{g}^*)$, and take advantage of a natural Poisson superalgebra structure on this space, defined as follows. The commutative superalgebra structure is given by the exterior multiplication, whereas the (even) Poisson bracket

$$\{\cdot, \cdot\} : \wedge^k(\mathfrak{g} \oplus \mathfrak{g}^*) \times \wedge^l(\mathfrak{g} \oplus \mathfrak{g}^*) \longrightarrow \wedge^{k+l-2}(\mathfrak{g} \oplus \mathfrak{g}^*)$$

(called *the big bracket* in [29] and [22], although it goes back to [27]) is uniquely determined by the following properties:

- For any $a, b \in \mathfrak{g}$, $\{a, b\} = 0$;
- For any $\xi, \eta \in \mathfrak{g}^*$, $\{\xi, \eta\} = 0$;
- For any $a \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$, $\{\xi, a\} = \xi(a)$;
- $\{\cdot, \cdot\}$ is skew-symmetric, i.e. for any $e_1 \in \wedge^k(\mathfrak{g} \oplus \mathfrak{g}^*)$, $e_2 \in \wedge^l(\mathfrak{g} \oplus \mathfrak{g}^*)$,

$$\{e_1, e_2\} = -(-1)^{kl} \{e_2, e_1\}$$

- For every $e \in \wedge^k(\mathfrak{g} \oplus \mathfrak{g}^*)$, $\{e, \cdot\}$ is a derivation of the exterior algebra $\wedge(\mathfrak{g} \oplus \mathfrak{g}^*)$ of degree $k - 2$.

In other words. $\{\cdot, \cdot\}$ is the unique extension of the canonical symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathfrak{g} \oplus \mathfrak{g}^*$ (2.2) to an even Poisson superalgebra structure on $\wedge(\mathfrak{g} \oplus \mathfrak{g}^*)$: it is easy to show that the super Jacobi identity

$$\{e_1, \{e_2, e_3\}\} = \{\{e_1, e_2\}, e_3\} + (-1)^{kl} \{e_2, \{e_1, e_3\}\}$$

holds for all $e_1 \in \wedge^k(\mathfrak{g} \oplus \mathfrak{g}^*)$, $e_2 \in \wedge^l(\mathfrak{g} \oplus \mathfrak{g}^*)$, $e_3 \in \wedge^m(\mathfrak{g} \oplus \mathfrak{g}^*)$.

Using this operation, it can be shown without difficulty that $(\mathfrak{g}, \mu, \gamma)$ is a Lie bialgebra if and only if

$$\{\mu, \mu\} = \{\gamma, \gamma\} = \{\mu, \gamma\} = 0. \quad (3.1)$$

Here $\{\mu, \mu\} = 0$ (resp. $\{\gamma, \gamma\} = 0$) is equivalent to the Jacobi identity for $[\cdot, \cdot]$ (resp. $[\cdot, \cdot]_*$), while $\{\mu, \gamma\} = 0$ is equivalent to the cocycle condition. The brackets $[\cdot, \cdot]$ and $[\cdot, \cdot]_*$ can be recovered by the formulas

$$\begin{aligned} [a, b] &= \{\{\mu, a\}, b\} \\ [\xi, \eta]_* &= \{\{\gamma, \xi\}, \eta\} \end{aligned} \quad (3.2)$$

where $a, b \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$. Furthermore, if we set $\theta = \mu + \gamma$, then (3.1) is equivalent to

$$\{\theta, \theta\} = 0, \quad (3.3)$$

and if $e_1, e_2 \in \mathfrak{g} \oplus \mathfrak{g}^* \subset \wedge(\mathfrak{g} \oplus \mathfrak{g}^*)$,

$$[e_1, e_2] = \{\{\theta, e_1\}, e_2\} \quad (3.4)$$

is precisely the Drinfeld double bracket on $\mathfrak{g} \oplus \mathfrak{g}^*$ (2.1): it is skew-symmetric, since the symmetric part is proportional to $\{\theta, \{e_1, e_2\}\}$ which is zero because $\{e_1, e_2\} \in \mathbb{R}$; it is easy to see from (3.2) that \mathfrak{g} and \mathfrak{g}^* are subalgebras, and that the canonical inner product $\langle \cdot, \cdot \rangle$ is ad-invariant; therefore, it must coincide with (2.1). The Jacobi identity is a consequence of (3.3).

From the point of view of [22], the main advantage of this approach is that it affords an elegant treatment of Drinfeld's *quasi-Lie bialgebras*, a generalization of Lie bialgebras in which the Jacobi identity for one of the brackets is satisfied only up to a coboundary. This amounts to adding a $\phi \in \wedge^3 \mathfrak{g}$ or $\psi \in \wedge^3 \mathfrak{g}^*$ (or both) to θ so that θ still satisfies (3.3). Note that the double (3.4) is still a Lie algebra, even though \mathfrak{g} or \mathfrak{g}^* are not.

For our purposes, however, the chief value of this approach is that it generalizes to Lie bialgebroids, if interpreted correctly; this will be our next order of business.

3.2 Lie algebroids revisited

As a naive attempt to generalize the above construction to Lie algebroids, we might try, given a vector bundle $A \rightarrow M$, to consider the exterior algebra $\Gamma(\wedge(A \oplus A^*))$ and build

the “big bracket” $\{\cdot, \cdot\}$ on this space from the canonical inner product $\langle \cdot, \cdot \rangle$ on $A \oplus A^*$ (see Example 2.6.7), as above. However, one quickly realizes that this is not enough to encode a Lie algebroid structure on A or A^* . First, the anchor: $a : A \rightarrow TM$ can be viewed as a section of $A^* \otimes TM$, so there is no room for it in $\Gamma(\bigwedge(A \oplus A^*))$; on the other hand, the Lie algebroid bracket (say, on $\Gamma(A)$) can no longer be viewed as a section of $\bigwedge^2 A^* \otimes A$, since it is not linear over $C^\infty(M)$ unless a is trivial. Furthermore, the structural identities of a Lie algebroid (e.g. the Jacobi identity) are not algebraic but differential equations, so they cannot be encoded by $\{\cdot, \cdot\}$ which is $C^\infty(M)$ -linear. Thus it is clear that we need a bigger space with an even Poisson superalgebra structure in which $(\Gamma(\bigwedge(A \oplus A^*)), \{\cdot, \cdot\})$ can be embedded. In order to find this space, we must shift our point of view from an algebraic to a geometric one and use the language of supermanifolds.

Recall from Chapter 2 that a Lie algebroid structure on a vector bundle A over M is equivalent to a derivation d_A of the exterior algebra $\Gamma(\bigwedge A^*)$ of degree one and square zero. Just as above, we will view $\Gamma(\bigwedge A^*)$ as the algebra of functions on the supermanifold ΠA , where Π here denotes the change of parity functor applied to each fibre (see Appendix). If $\{x^i\}_{i=1, \dots, \dim M}$ is a coordinate chart on $U \subset M$, and $\{e^a\}_{a=1, \dots, \text{rk} A}$ is a local basis of sections of A^* over U (dual to a basis $\{e_a\}$ of sections of A), denote by ξ^a the corresponding generators of the Grassman algebra $\Gamma(U, \bigwedge A^*)$; then $\{(x^i, \xi^a)\}$ give a coordinate chart on ΠA with the transformation law inherited from the vector bundle A^* . The derivation d_A can then be viewed as an (odd) vector field on ΠA , satisfying

$$[d_A, d_A] = 2d_A^2 = 0, \quad (3.5)$$

where the bracket denotes the (super)commutator. Such vector fields are called *homological* for an obvious reason. This motivates the following

Definition 3.2.1. A Lie algebroid structure on a vector bundle $A \rightarrow M$ is the supermanifold ΠA together with a homological vector field d_A of degree 1.

Remark 3.2.2. This interpretation of Lie algebroids was proposed by Kontsevich [20] and Vaintrob [39]. It is important that d_A be of degree 1 with respect to the natural \mathbb{Z} -grading on functions on ΠA , rather than merely be odd, in order to define a Lie algebroid structure on A . Arbitrary odd homological vector fields on supermanifolds lead to strongly homotopy Lie algebras [3].

In local coordinates, we have, according to the Cartan formula (2.3),

$$d_A = \xi^a A_a^i(x) \frac{\partial}{\partial x^i} - \frac{1}{2} C_{ab}^c(x) \xi^a \xi^b \frac{\partial}{\partial \xi^c} \quad (3.6)$$

where

$$\begin{aligned} a(e_a) &= A_a^i(x) \frac{\partial}{\partial x^i} \\ [e_a, e_b]_A &= C_{ab}^c(x) e_c \end{aligned}$$

are the local expressions for the anchor and the bracket on the Lie algebroid A . Similarly, a Lie algebroid structure on the dual bundle A^* is equivalent to a homological vector field d_{A^*} on the supermanifold ΠA^* given in local coordinates (x^i, θ_a) by

$$d_{A^*} = \theta_a \bar{A}^{ai}(x) \frac{\partial}{\partial x^i} - \frac{1}{2} \bar{C}_c^{ab}(x) \theta_a \theta_b \frac{\partial}{\partial \theta^c} \quad (3.7)$$

where

$$\begin{aligned} a_*(e^a) &= \bar{A}^{ai}(x) \frac{\partial}{\partial x^i} \\ [e^a, e^b]_{A^*} &= \bar{C}_c^{ab}(x) e^c \end{aligned}$$

are the local expressions for the anchor and the bracket on A^* .

3.3 The cotangent bundle

Once in a “supermathematical” frame of mind, one immediately realizes that the exterior algebra $\wedge(\mathfrak{g} \oplus \mathfrak{g}^*)$ is to be interpreted as the algebra of functions on the (purely odd) superspace $\Pi(\mathfrak{g} \oplus \mathfrak{g}^*)$. The crucial observation, however, is that $\Pi(\mathfrak{g} \oplus \mathfrak{g}^*)$ is naturally isomorphic to the cotangent bundle $T^*\Pi\mathfrak{g}$, while the big bracket $\{\cdot, \cdot\}$ is nothing but the canonical symplectic Poisson bracket on $T^*\Pi\mathfrak{g}$. Indeed, if $\{e_a\}_{a=1, \dots, \dim \mathfrak{g}}$ is a basis of \mathfrak{g} , $\{e^a\}$ the dual basis, denote by $\{\theta_a\}$ the corresponding generators of the Grassman algebra $\wedge \mathfrak{g} = C^\infty(\Pi\mathfrak{g}^*)$, and $\{\xi^a\}$ the corresponding generators of $\wedge \mathfrak{g}^* = C^\infty(\Pi\mathfrak{g})$. Then the θ_a can be viewed as the momenta conjugate to the Grassman coordinates ξ^a on $\Pi\mathfrak{g}$, and the defining relations of the big bracket (Section 3.1) can be rewritten as

$$\{\xi^a, \xi^b\} = 0; \quad \{\theta_a, \theta_b\} = 0; \quad \{\xi^a, \theta_b\} = \delta_b^a$$

which one immediately recognizes as the canonical Poisson brackets between coordinates and momenta on $T^*\Pi\mathfrak{g}$. This bracket is nondegenerate and even, in the sense that the bracket

of two Grassman polynomials of parity $\epsilon_1, \epsilon_2 \in \mathbb{Z}_2$ is of parity $\epsilon_1 + \epsilon_2$; the corresponding symplectic 2-form is

$$\omega = d\theta_a d\xi^a$$

Thus, $T^*\Pi\mathfrak{g}$ is an even symplectic supermanifold (see Appendix). This is completely analogous to the canonical symplectic structure on $T^*V \simeq V \oplus V^*$, where V is a vector space, except now the Poisson bracket on linear functions (which are odd) is symmetric rather than skew-symmetric; in fact, the matrix of ω coincides with the matrix of $\langle \cdot, \cdot \rangle$ (2.2) in the basis $\{e_a, e^b\}$ of $\mathfrak{g} \oplus \mathfrak{g}^*$. The advantage of this point of view is that it generalizes immediately to vector bundles.

Just as in the purely odd or even case, given any supermanifold Q , its cotangent bundle T^*Q is an even symplectic supermanifold. If $\{x^\alpha\}$ is a coordinate chart for Q , the corresponding Darboux chart for T^*Q is $\{x^\alpha, x_\alpha^*\}$, where x_α^* is of the same parity as x^α and the canonical Poisson brackets are given by

$$\{x^\alpha, x^\beta\} = 0; \quad \{x_\alpha^*, x_\beta^*\} = 0; \quad \{x_\alpha^*, x^\beta\} = \delta_\alpha^\beta \quad (3.8)$$

Any vector field v on a Q gives rise to a fibrewise-linear function h_v on the cotangent bundle T^*Q in an obvious manner: in local coordinates, if $v = v^\alpha(x) \frac{\partial}{\partial x^\alpha}$, then $h_v = v^\alpha(x) x_\alpha^*$ in the corresponding Darboux coordinates on T^*Q . This is well defined since the momenta x_α^* transform in the same way as the derivations $\frac{\partial}{\partial x^\alpha}$ under changes of coordinates on Q . This “hamiltonian lift” has the following properties:

Lemma 3.3.1. *Let v, w be vector fields on Q , $f \in C^\infty(Q)$, and let $\pi : T^*Q \rightarrow Q$ denote the canonical projection. Then*

1. $\{h_v, \pi^* f\} = \pi^*(vf)$
2. $\{h_v, h_w\} = h_{[v, w]}$

Proof. This is best done by direct computation in local coordinates, just as for ordinary manifolds. (1) is obvious by (3.8) and the definition of h_v , whereas for (2) we can easily deduce from (3.8) that

$$\begin{aligned} \{h_v, h_w\} &= \{v^\alpha(x) x_\alpha^*, w^\beta(x) x_\beta^*\} = \\ &= (v^\alpha \frac{\partial w^\beta}{\partial x^\alpha} - (-1)^{\tilde{v}\tilde{w}} w^\alpha \frac{\partial v^\beta}{\partial x^\alpha}) x_\beta^* = h_{[v, w]}, \end{aligned}$$

where \tilde{v} denotes the parity of the vector field v . □

Now let $Q = \Pi A$, and let $\mu = h_{d_A} \in C^\infty(T^*\Pi A)$. Then by Lemma 3.3.1 and (3.5) we immediately get

$$\{\mu, \mu\} = 0 \quad (3.9)$$

The formula (3.6) leads to the following local expression for μ :

$$\mu = \xi^a A_a^i(x) x_i^* - \frac{1}{2} C_{ab}^c(x) \xi^a \xi^b \xi_c^* \quad (3.10)$$

Thus, a Lie algebroid structure on A gives rise to an odd linear function μ on $T^*\Pi A$ satisfying $\{\mu, \mu\} = 0$, but how do we characterize those that come from Lie algebroids? Two remarks are in order.

Remark 3.3.2. Unless the bundle $A \rightarrow M$ is trivial, the supermanifold $T^*\Pi A$ is *not* of the form ΠV in any natural way, where V is some vector bundle over T^*M , the support (“even part”) of $T^*\Pi A$. That is to say, there is no canonical projection from $T^*\Pi A$ to T^*M and a natural \mathbb{Z}_+ -grading on $C^\infty(T^*\Pi A)$ inducing the \mathbb{Z}_2 -grading and respecting the projection. The reason is that under the natural fiberwise linear coordinate changes

$$\begin{aligned} x^i &= x^i(x') \\ \xi^a &= T_{a'}^a(x') \xi^{a'} \end{aligned}$$

on ΠA the momenta transform by

$$\begin{aligned} x_i^* &= \frac{\partial x^{i'}}{\partial x^i}(x(x')) x_{i'}^* + \frac{\partial T_{b'}^{a'}}{\partial x^i}(x(x')) T_{b'}^b(x') \xi^{b'} \xi_{a'}^* \\ \xi_a^* &= T_a^{a'}(x(x')) \xi_{a'}^* \end{aligned}$$

so the total degree in the odd variables (ξ^a, ξ_b^*) is not preserved because of the second term in the transformation law for x_i^* . Because of this fact, our constructions cannot be recast in the “classical” framework of manifolds and vector bundles - one cannot get around using supermanifolds.

Remark 3.3.3. Nevertheless, there are three \mathbb{Z}_+ -gradings on fiberwise-polynomial functions on $T^*\Pi A$ which are preserved under the natural transformations above. The first one, which we denote by ϵ , exists by virtue of the fact that $T^*\Pi A$ is a vector bundle over ΠA - it is simply the fiberwise degree, i.e the total degree of a polynomial in the momenta (x_i^*, ξ_a^*) ; the second, δ , is the total degree of a polynomial in (x_i^*, ξ^a) . These gradings are not compatible with the \mathbb{Z}_2 -grading (parity), since the even variables x_i^* have $\epsilon(x_i^*) = \delta(x_i^*) = 1$, but their sum κ is. The total grading κ assigns degree 2 to x_i^* and degree 1 to ξ^a and ξ_a^* . Those

functions μ that come from Lie algebroid structures on A can be characterized by their (ϵ, δ) -bidegree, namely,

$$\epsilon(\mu) = 1; \delta(\mu) = 2; \kappa(\mu) = 3$$

The Poisson bracket $\{\cdot, \cdot\}$ has (ϵ, δ) -bidegree $(-1, -1)$, and hence total κ -degree -2 .

Similarly, a Lie algebroid structure on the dual bundle A^* gives rise to a linear function $\gamma = h_{d_{A^*}}$ on $T^*\Pi A^*$ satisfying $\{\gamma, \gamma\} = 0$. By (3.7), it is given in local coordinates $(x^i, \theta_a, x_i^*, \theta_*^a)$ by

$$\gamma = \theta_a \bar{A}^{ai}(x) x_i^* - \frac{1}{2} \bar{C}_c^{ab}(x) \theta_a \theta_b \theta_*^c \quad (3.11)$$

Since *a priori* the functions μ and γ live on different manifolds, it seems unclear at this point how to express the compatibility condition between them in case (A, A^*) is a Lie bialgebroid. It is also not clear how to express the Schouten brackets $[\cdot, \cdot]_A$ and $[\cdot, \cdot]_{A^*}$ in this formalism. Fortunately, it turns out that the supermanifolds $T^*\Pi A$ and $T^*\Pi A^*$ are canonically symplectomorphic, via the *Legendre transform*.

3.4 The Legendre transform

The Legendre transform is widely known in classical mechanics for its crucial role in providing a transition from the Lagrangian to the Hamiltonian formalism. Recall that if M is the configuration space of a classical mechanical system, $l = l(q, \dot{q}) \in C^\infty(TM)$ the Lagrangian function, then the dynamics of the system are given by the *Euler-Lagrange equations*

$$\frac{\partial l}{\partial q^i}(q(t), \dot{q}(t)) - \frac{d}{dt} \frac{\partial l}{\partial \dot{q}^i}(q(t), \dot{q}(t)) = 0$$

satisfied by a classical trajectory $q = q(t)$. Then one introduces the momenta $p_i \in C^\infty(T^*M)$ by

$$p_i = \frac{\partial l}{\partial \dot{q}^i}(q, \dot{q})$$

Suppose the Lagrangian l is strongly nondegenerate in the sense that the above equations have a unique solution $\dot{q}^i = \dot{q}^i(q, p)$. Then one can define the Hamiltonian function $h = h(q, p) \in C^\infty(T^*M)$ as the Legendre transform of l , i.e.

$$h(q, p) = \dot{q}^i p_i - l(q, \dot{q})$$

where we substitute $\dot{q}^i = \dot{q}^i(q, p)$. The Euler-Lagrange equations are equivalent to *Hamilton's equations*

$$\frac{dq^i}{dt} = \frac{\partial h}{\partial p_i}; \quad \frac{dp_i}{dt} = -\frac{\partial h}{\partial q^i}$$

In 1977, W.M. Tulczyjew [38] gave a geometric interpretation of the Legendre transform as a canonical symplectomorphism between the cotangent bundles $T^*(TM)$ and $T^*(T^*M)$. It turns out that in Tulczyjew's construction one can replace TM with an arbitrary vector bundle A or a supermanifold ΠA . We shall now describe this construction.

Let P be manifold, $Q \subset P$ a submanifold, $f \in C^\infty(Q)$. This data gives rise to a Lagrangian submanifold of $L \subset T^*P$ as follows:

$$L = \{\xi \in T^*P \mid \pi_P(\xi) = x \in Q; \xi(v) = df(v) \forall v \in T_x Q\},$$

where π_P denotes the canonical projection $T^*P \rightarrow P$. If $f = 0$, L is just the conormal bundle to Q ; if $Q = P$, L is just the image of df .

We are interested in the following special case. Let A be the total space of a vector bundle $A \rightarrow M$, A^* that of the dual bundle. Consider their fibre product, i.e. the total space of the Whitney sum $Q = A \oplus A^* \subset A \times A^* = P$. On Q , there is a canonical evaluation function $f = -ev \in C^\infty(Q)$ given by $f(v, \xi) = -v(\xi)$. This gives rise to a Lagrangian submanifold $L \subset T^*(A \times A^*) \simeq T^*A \times \overline{T^*A^*}$, where the bar denotes the opposite symplectic structure and the isomorphism is given by the ‘‘Schwartz transform’’

$$S((x, y), (\xi, \eta)) = ((x, \xi), (y, -\eta))$$

(see [8] for an explanation of this name). This $L \subset T^*A \times \overline{T^*A^*}$ is the graph of a symplectomorphism $L : T^*A \rightarrow T^*A^*$ that can be interpreted as a geometric version of the Legendre transform. It is given in local coordinates simply by

$$((x, v), (p, \xi)) \longmapsto ((x, -\xi), (p, v))$$

Example 3.4.1. Let $A = TM$, $l \in C^\infty(TM)$ a strongly nondegenerate Lagrangian. Then the image of TM under $-dl : TM \rightarrow T^*TM$ followed by $L : T^*(TM) \rightarrow T^*(T^*M)$ coincides with the image of $dh : T^*M \rightarrow T^*(T^*M)$, where h is the classical Legendre transform of l .

What is important for our purposes is that all of the above carries over to supermanifolds without change, as long as the function f is even, otherwise df is a section not of

T^*Q but of ΠT^*Q . On $\Pi(A \oplus A^*)$ there is a canonical even function $ev \in C^\infty(\Pi(A \oplus A^*))$ given in local coordinates by

$$ev(x, \xi, \theta) = \xi^a \theta_a,$$

giving rise to a canonical symplectomorphism $L : T^*\Pi A \rightarrow T^*\Pi A^*$ via the above construction. In local coordinates,

$$L(x, \xi, x^*, \xi^*) = (x, \xi^*, x^*, \xi) \quad (3.12)$$

In other words, the fibre coordinates ξ^a on ΠA become conjugate to the fibre coordinates θ_a on ΠA^* , and vice versa. In a way, this local description is more illuminating than the canonical geometric construction above, but we choose to present the geometric construction rather than go through a proof that (3.12) does not depend on a choice of local coordinates.

Example 3.4.2. Consider the supermanifold ΠTM and a 2-form ω on M viewed as a quadratic function on ΠTM . If ω is nondegenerate, then the image of ΠTM under $d\omega : \Pi TM \rightarrow T^*\Pi TM$ followed by the Legendre transform $L : T^*\Pi TM \rightarrow T^*\Pi T^*M$ coincides with the image of $d\pi : \Pi T^*M \rightarrow T^*\Pi T^*M$, where $\pi \in C^\infty(\Pi T^*M)$ is the bivector field given by the inverse of ω . Indeed, if $\omega = \frac{1}{2}\omega_{ab}(x)\xi^a\xi^b$, then the image of $d\omega$ in $T^*\Pi TM$ is given by

$$\begin{aligned} x_c^* &= \frac{\partial \omega}{\partial x^c} = \frac{1}{2} \frac{\partial \omega_{ab}}{\partial x^c} \xi^a \xi^b \\ \xi_c^* &= \frac{\partial \omega}{\partial \xi^c} = \omega_{cb}(x) \xi^b \end{aligned}$$

The second equation can be solved for ξ if and only if ω is nondegenerate, in which case

$$\xi^a = \pi^{ab}(x) \xi_b^*$$

where $\pi^{ab}(x)\omega_{bc}(x) = \delta_c^a$. Applying the Legendre transform (3.12), we get

$$\begin{aligned} x_c^* &= \frac{1}{2} \frac{\partial \omega_{ab}}{\partial x^c} \theta_*^a \theta_*^b \\ \theta_c &= \omega_{cb}(x) \theta_*^b \end{aligned}$$

and

$$\theta_*^a = \pi^{ab}(x) \theta_b$$

Setting

$$-\pi(x, \theta) = \xi^a \theta_a - \omega(x, \xi) = -\frac{1}{2} \pi^{ab}(x) \theta_a \theta_b$$

we get

$$\begin{aligned} x_c^* &= \frac{\partial \pi}{\partial x^c} \\ \theta_*^c &= \frac{\partial \pi}{\partial \theta_c} \end{aligned}$$

Note that here d denotes the deRham differential of functions on $\Pi T M$ (or $\Pi T^* M$), not of forms on M ! Notice also how this example parallels Example 3.4.1.

Let us now derive some properties of the Legendre transform that will be useful in what follows. We begin by drawing the following diagram:

$$\begin{array}{ccc} T^*\Pi A & \xrightarrow{L} & T^*\Pi A^* \\ \downarrow \pi & & \downarrow \bar{\pi} \\ \Pi A & & \Pi A^* \end{array} \quad (3.13)$$

where π and $\bar{\pi}$ are the canonical projections. It is obvious that $\pi_A \circ \pi = \pi_{A^*} \circ \bar{\pi} \circ L$, where $\pi_A : \Pi A \rightarrow M$ and $\pi_{A^*} : \Pi A^* \rightarrow M$ are the canonical projections; therefore, by abstract nonsense the diagram above gives rise to a projection to the fibered product

$$T^*\Pi A \xrightarrow{p} \Pi(A \oplus A^*) \quad (3.14)$$

More specifically, if $\xi \in \Gamma(A^*)$ viewed as a linear function on ΠA or on $\Pi(A \oplus A^*)$, then $p^*\xi = \pi^*\xi$; on the other hand, ξ also gives rise to a vector field i_ξ on ΠA^* , the interior derivative, hence a linear hamiltonian h_{i_ξ} on $T^*\Pi A^*$. Similarly, an $X \in \Gamma(A)$ can be viewed as either a linear function on ΠA^* , pulled back to $T^*\Pi A^*$, or a vector field i_X on ΠA lifted to the hamiltonian h_{i_X} on $T^*\Pi A$. These functions are related by the Legendre transform, according to the following

Lemma 3.4.3. *For $X \in \Gamma(A)$, $\xi \in \Gamma(A^*)$,*

$$\begin{aligned} L^*\bar{\pi}^*X &= h_{i_X} \\ L^*h_{i_\xi} &= \pi^*\xi \end{aligned}$$

Proof. If locally $X = X^a(x)\theta_a$, $\xi = f_a(x)\xi^a$, then $i_X = X^a(x)\frac{\partial}{\partial \xi^a}$, hence

$$h_{i_X} = X^a(x)\xi_a^* = L^*(X^a(x)\theta_a) = L^*\bar{\pi}^*X$$

by (3.12), and similarly for ξ . □

In particular, $p^*\xi = \pi^*\xi$, $p^*X = L^*\bar{\pi}^*X = h_{i_X}$ and, of course, if $f \in C^\infty(M)$, $p^*f = \pi^*\pi_A^*f = \bar{\pi}^*\pi_{A^*}^*f$. This can be interpreted as follows. On $\Pi(A \oplus A^*)$ there is a natural even Poisson structure given by the canonical inner product on $A \oplus A^*$. The corresponding Poisson bracket is just the fiberwise big bracket described in the beginning of this section. The symplectic leaves are the fibres that inherit the big bracket, so the pullbacks of functions on M are the Casimir functions. We have the following

Corollary 3.4.4. *The projection p (3.14) is a Poisson map.*

Proof. Immediate from (2.4) and Lemmas 3.3.1 and 3.4.3. □

In other words, $T^*\Pi A$ is a symplectic realization of the Poisson supermanifold $\Pi(A \oplus A^*)$. This fact will be useful in dealing with Courant algebroids. But for now, we need one more construction to be able to deal with the Schouten brackets and Lie bialgebroids.

3.5 Derived brackets

Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, d)$ be an even or odd differential Lie superalgebra. That is,

- $\mathcal{A} = \mathcal{A}_0 \oplus \mathcal{A}_1$ is a \mathbb{Z}_2 -graded vector space;
- $[\cdot, \cdot]_{\mathcal{A}}$ is skew-symmetric of parity $\epsilon \in \mathbb{Z}_2$, i.e. $[\mathcal{A}_i, \mathcal{A}_j]_{\mathcal{A}} \subset \mathcal{A}_{i+j+\epsilon}$ and

$$[a, b]_{\mathcal{A}} = -(-1)^{(\tilde{a}+\epsilon)(\tilde{b}+\epsilon)}[b, a]_{\mathcal{A}}$$

for all $a \in \mathcal{A}_{\tilde{a}}$, $b \in \mathcal{A}_{\tilde{b}}$;

- $[\cdot, \cdot]_{\mathcal{A}}$ satisfies the Jacobi identity

$$[a, [b, c]_{\mathcal{A}}]_{\mathcal{A}} = [[a, b]_{\mathcal{A}}, c]_{\mathcal{A}} + (-1)^{(\tilde{a}+\epsilon)(\tilde{b}+\epsilon)}[b, [a, c]_{\mathcal{A}}]_{\mathcal{A}}$$

- $d : \mathcal{A} \rightarrow \mathcal{A}$ is an odd derivation of $[\cdot, \cdot]_{\mathcal{A}}$:

$$d[a, b]_{\mathcal{A}} = [da, b]_{\mathcal{A}} + (-1)^{\tilde{a}+\epsilon}[a, db]_{\mathcal{A}}$$

satisfying $d^2 = 0$.

One defines the *derived bracket* on \mathcal{A} as follows:

$$a \circ_d b = [ad_a, d]b = (-1)^{\tilde{a}+1}[da, b]_{\mathcal{A}}$$

where the bracket in the middle is the (super)commutator of derivations of \mathcal{A} and $ad_a = [a, \cdot]_{\mathcal{A}}$. The derived bracket has parity $\epsilon + 1$. It is not necessarily skew-symmetric; its skew-symmetrization

$$[a, b]_d = \frac{1}{2}(a \circ_d b - (-1)^{(\tilde{a}+1)(\tilde{b}+1)}b \circ_d a)$$

is also sometimes called the derived bracket, but the non skew-symmetric version is, in some sense, more fundamental and enjoys many nice properties.¹ They are summarized in the following

Lemma 3.5.1. *The derived bracket has the following properties:*

1. $a \circ_d (b \circ_d c) = (a \circ_d b) \circ_d c + (-1)^{(\tilde{a}+\epsilon+1)(\tilde{b}+\epsilon+1)}b \circ_d (a \circ_d c) \quad \forall a, b, c \in \mathcal{A}.$
2. $d(a \circ_d b) = (da) \circ_d b + (-1)^{\tilde{a}+\epsilon+1}a \circ_d (db) \quad \forall a, b \in \mathcal{A}.$
3. $a \circ_d b = [a, b]_d + \frac{(-1)^{\tilde{a}+\epsilon+1}}{2}d[a, b]_{\mathcal{A}}$

Proof. A straightforward computation, carried out in [25]. Property 1 depends both on the Jacobi identity for $[\cdot, \cdot]_{\mathcal{A}}$ and $d^2 = 0$. \square

The first two properties imply that $(\mathcal{A}, \circ_d, d)$ is a *differential Leibniz superalgebra* in the sense of Loday [32] (they are called *Loday algebras* in [25]). The third property implies that \circ_d is skew-symmetric up to a d -coboundary. Notice how these properties resemble some of the properties of Courant algebroids (Definition 2.6.1 and (2.16)).

Corollary 3.5.2. *Let $\mathcal{B} \subset \mathcal{A}$ be an abelian subalgebra (with respect to $[\cdot, \cdot]_{\mathcal{A}}$) closed under \circ_d ; then the restriction of \circ_d to \mathcal{B} is skew-symmetric and $(\mathcal{B}, [\cdot, \cdot]_d)$ is a Lie superalgebra of parity $\epsilon + 1$; if, moreover, $d\mathcal{B} \subset \mathcal{B}$, then $(\mathcal{B}, [\cdot, \cdot]_d, d)$ is a differential Lie superalgebra.*

Remark. Sometimes \mathcal{A} also has a (super)commutative algebra structure such that $[\cdot, \cdot]_{\mathcal{A}}$ is a derivation of the multiplication in each argument. If $\epsilon = 0$ (i.e. \mathcal{A} is an even Poisson superalgebra), then \mathcal{B} becomes an odd Poisson superalgebra, i.e. a Gerstenhaber algebra. On the other hand, if \mathcal{A} is a Gerstenhaber algebra, \mathcal{B} is an even Poisson superalgebra [25].

¹ It is somewhat unfortunate that what we call the derived bracket is not denoted by a bracket; nevertheless, we feel that the bracket notation ought to be reserved for skew-symmetric operations. In [25] f_d was used to denote the derived bracket regardless of skew-symmetry.

Example 3.5.3. Consider $\mathcal{A} = C^\infty(\Pi T^*M) = \Gamma(\wedge TM) = \mathfrak{X}^*(M)$, the multivector fields on a supermanifold M endowed with the Schouten bracket $[\cdot, \cdot]$; pick a quadratic function (a bivector field) π satisfying $[\pi, \pi] = 0$ and consider the inner derivation $d_\pi = [\pi, \cdot]$; \mathcal{A} is a differential Gerstenhaber algebra. Then $\mathcal{B} = C^\infty(M)$ is an abelian subalgebra of \mathcal{A} stable under d_π , and the derived bracket on \mathcal{B}

$$\{f, g\} = (-1)^{\tilde{f}} [[\pi, f], g]$$

is precisely the Poisson bracket generated by the bivector field π .

Example 3.5.4. If $(\mathfrak{g}, \mu, \gamma)$ is a Lie bialgebra, let $\mathcal{A} = \wedge(\mathfrak{g} \oplus \mathfrak{g}^*)$ with $[\cdot, \cdot]_{\mathcal{A}} = \{\cdot, \cdot\}$, the big bracket. Then $\wedge \mathfrak{g}$ is an abelian subalgebra stable under the differential $\{\mu, \cdot\}$, while $\wedge \mathfrak{g}^*$ is an abelian subalgebra stable under $\{\gamma, \cdot\}$. The corresponding derived brackets give the *algebraic Schouten brackets*, generalizing the formulas (3.2) [22]. Notice how the Drinfeld double bracket (3.4) is generated by $\theta = \mu + \gamma$ as a derived bracket: although $\mathfrak{g} \oplus \mathfrak{g}^*$ is not closed under $\{\cdot, \cdot\}$, it is closed under the derived bracket.

We will show below that the Schouten brackets associated to Lie algebroids, as well as the Courant bracket (2.19), arise in exactly the same way.

3.6 Schouten brackets, Lie bialgebroids and the Drinfeld double

The concept of a derived bracket enables us to define the Schouten brackets and recast the notion of Lie bialgebroid in the supermanifold context. Let d_{A^*} be a homological vector field on ΠA^* giving rise to a Lie algebroid structure on $A^* \rightarrow M$; let $\gamma = h_{d_{A^*}}$ be the corresponding linear hamiltonian on $T^*\Pi A^*$, and consider its Legendre transform $L^*\gamma \in C^\infty(T^*\Pi A)$. By (3.11) and (3.12),

$$L^*\gamma = \bar{A}^{ai}(x)x_i^*\xi_a^* - \frac{1}{2}\xi^c\bar{C}_c^{ab}(x)\xi_a^*\xi_b^* \quad (3.15)$$

Remark 3.6.1. Notice that $L^*\gamma$ is fiberwise quadratic, i.e. $\epsilon(L^*\gamma) = 2$; on the other hand, $\delta(L^*\gamma) = 1$, so the total degree $\kappa(L^*\gamma)$ is again 3. This characterizes those functions on $T^*\Pi A$ that come from Lie algebroid structures on A^* . In fact, the grading δ is seen to correspond to the momentum grading ϵ^* on $T^*\Pi A^*$ under L , whereas the ϵ -grading

corresponds to δ^* . Thus, the Legendre transform interchanges the ϵ and δ gradings and preserves the total grading κ .

Since L is a symplectomorphism, we have

$$\{L^*\gamma, L^*\gamma\} = L^*\{\gamma, \gamma\} = 0,$$

hence $(C^\infty(T^*\Pi A), \{\cdot, \cdot\}, \{L^*\gamma, \cdot\})$ is a differential Lie superalgebra, and we can consider the derived bracket. It turns out that the abelian subalgebra $\pi^*C^\infty(\Pi A)$ is closed under the derived bracket, and the restriction of the derived bracket coincides with the Schouten bracket $[\cdot, \cdot]_{A^*}$. More precisely, we have

Lemma 3.6.2. *Let $\xi, \eta \in C^\infty(\Pi A) = \Gamma(\bigwedge A^*)$. Then*

$$\pi^*[\xi, \eta]_{A^*} = (-1)^{\tilde{\xi}+1} \{\{L^*\gamma, \pi^*\xi\}, \pi^*\eta\}$$

Proof. The skew-symmetry and derivation property are consequences of Corollary 3.5.2. Hence, we only need to consider fiberwise constant and fiberwise linear functions, i.e. elements of $C^\infty(M)$ and $\Gamma(A^*)$. We have:

$$\begin{aligned} \{\{L^*\gamma, \pi^*f\}, \pi^*g\} &= \{L^*\{\gamma, \bar{\pi}^*f\}, \pi^*g\} = \{L^*\{h_{d_{A^*}}, \bar{\pi}^*f\}, \pi^*g\} = \\ &= \{L^*\bar{\pi}^*d_{A^*}f, \pi^*g\} = L^*\{\bar{\pi}^*d_{A^*}f, \bar{\pi}^*g\} = 0 = \pi^*[f, g]_{A^*} \end{aligned}$$

for all $f, g \in C^\infty(M)$;

$$\begin{aligned} \{\{L^*\gamma, \pi^*\xi\}, \pi^*f\} &= \{L^*\{h_{d_{A^*}}, h_{i_\xi}\}, \pi^*f\} = L^*\{h_{[d_{A^*}, i_\xi]}, \bar{\pi}^*f\} = \\ &= L^*\bar{\pi}^*(L_\xi^{A^*}f) = \pi^*(a_*(\xi)f) = \pi^*[\xi, f]_{A^*} \end{aligned}$$

for all $f \in C^\infty(M)$, $\xi \in \Gamma(A^*)$. And finally,

$$\begin{aligned} \{\{L^*\gamma, \pi^*\xi\}, \pi^*\eta\} &= \{L^*\{h_{d_{A^*}}, h_{i_\xi}\}, \pi^*\eta\} = L^*\{h_{[d_{A^*}, i_\xi]}, h_{i_\eta}\} = \\ &= L^*h_{[[d_{A^*}, i_\xi], i_\eta]} = L^*h_{[L_\xi^{A^*}, i_\eta]} = L^*h_{i_{[\xi, \eta]_{A^*}}} = \pi^*[\xi, \eta]_{A^*} \end{aligned}$$

for all $\xi, \eta \in \Gamma(A^*)$. We have made repeated use of the commutation relations (2.4), Lemma 3.3.1 and Lemma 3.4.3. \square

Remark. Of course, the same is true for the Schouten bracket $[\cdot, \cdot]_A$ associated to a Lie algebroid structure on A , if we use $(L^{-1})^*\mu$ where $\mu = h_{d_A}$.

Example 3.6.3. For any (super)manifold M , a canonical Lie algebroid structure on the bundle $A^* = TM$ (Example 2.2.4) is given by the de Rham differential d on ΠTM . The corresponding quadratic hamiltonian

$$L^*h_d = \theta_a^* x_a^*$$

on $T^*\Pi T^*M$ generates the Schouten bracket of multivector fields on M as the derived bracket.

Remark 3.6.4. The Schouten bracket $[\cdot, \cdot]_{A^*}$ on the supermanifold ΠA is an odd Poisson structure (see Appendix). Examples 3.5.3, 3.5.4 and 3.6.3 are special cases of the following general phenomenon: even Poisson structures on a supermanifold M are generated by bivector fields, i.e. even quadratic hamiltonians on the odd symplectic supermanifold ΠT^*M , whereas odd Poisson structures are generated by odd quadratic hamiltonians on the even symplectic supermanifold T^*M (see the Appendix in [42], also [25]).

We can now prove the following simple characterization of Lie bialgebroids.

Proposition 3.6.5. *A pair (A, A^*) of Lie algebroids in duality is a Lie bialgebroid if and only if*

$$\{\mu, L^*\gamma\} = 0, \quad (3.16)$$

where $\mu = h_{d_A}$, $\gamma = h_{d_{A^*}}$.

Proof. We must show that d_A is a derivation of $[\cdot, \cdot]_{A^*}$ if and only if (3.16) holds. However, by Lemma 3.3.1 and Lemma 3.6.2, we have

$$\begin{aligned} \pi^*d_A[\xi, \eta]_{A^*} &= \{\mu, \pi^*[\xi, \eta]_{A^*}\} = (-1)^{\tilde{\xi}+1}\{\mu, \{\{L^*\gamma, \pi^*\xi\}, \pi^*\eta\}\} \\ &= (-1)^{\tilde{\xi}+1}(\{\{\mu, \{L^*\gamma, \pi^*\xi\}\}, \pi^*\eta\} + \\ &+ (-1)^{\tilde{\xi}+1}\{\{L^*\gamma, \pi^*\xi\}, \{\mu, \pi^*\eta\}\}) = \\ &= (-1)^{\tilde{\xi}+1}(\{\{\{\mu, L^*\gamma\}, \pi^*\xi\}, \pi^*\eta\} - \\ &- \{\{L^*\gamma, \{\mu, \pi^*\xi\}\}, \pi^*\eta\} + (-1)^{\tilde{\xi}+1}\{\{L^*\gamma, \pi^*\xi\}, \pi^*d_A\eta\}) = \\ &= (-1)^{\tilde{\xi}}\{\{L^*\gamma, \pi^*d_A\xi\}, \pi^*\eta\} + \{\{L^*\gamma, \pi^*\xi\}, \pi^*d_A\eta\} + \\ &+ (-1)^{\tilde{\xi}+1}\{\{\{\mu, L^*\gamma\}, \pi^*\xi\}, \pi^*\eta\} = \\ &= \pi^*([d_A\xi, \eta]_{A^*} + (-1)^{\tilde{\xi}+1}[\xi, d_A\eta]_{A^*}) + \\ &+ (-1)^{\tilde{\xi}+1}\{\{\{\mu, L^*\gamma\}, \pi^*\xi\}, \pi^*\eta\} \end{aligned}$$

Since $\{\mu, L^*\gamma\}$ is fiberwise quadratic, the second term in the last expression vanishes if and only if $\{\mu, L^*\gamma\} = 0$. The statement follows by the injectivity of π^* . \square

Corollary 3.6.6. (A, A^*) is a Lie bialgebroid if and only if (A^*, A) is.

Proof. The Legendre transform L is a symplectomorphism. \square

Now set $\theta = \mu + L^*\gamma$. Clearly, (A, A^*) is a Lie bialgebroid if and only if

$$\{\theta, \theta\} = 0 \quad (3.17)$$

This motivates the following

Definition 3.6.7. Given a Lie bialgebroid (A, A^*) , its *Drinfeld double* is $T^*\Pi A$ together with the homological vector field $D = \{\theta, \cdot\}$.

Example 3.6.8. If $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, $C^\infty(T^*\Pi \mathfrak{g}) = \bigwedge(\mathfrak{g} \oplus \mathfrak{g}^*)^*$ and D is the Chevalley-Eilenberg differential in the standard complex of the Drinfeld double Lie algebra $\mathfrak{g} \oplus \mathfrak{g}^*$ (2.1).

Example 3.6.9. Let $A = M \times \mathfrak{g}$ be the action Lie algebroid corresponding to a Lie algebra action $\rho : \mathfrak{g} \rightarrow \mathfrak{X}(M)$ (Example 2.2.5). View (A, A^*) as a Lie bialgebroid with the trivial structure on A^* . Then $C^\infty(T^*\Pi A) = C^\infty(M) \otimes \bigwedge(\mathfrak{g} \oplus \mathfrak{g}^*)$, and $D = \{\mu, \cdot\}$ coincides with the classical BRST differential associated to the hamiltonian lift of ρ to T^*M . Indeed, recall that the classical BRST differential d is the sum of the Chevalley-Eilenberg differential δ for the Lie algebra \mathfrak{g} with values in the module $\bigwedge \mathfrak{g} \otimes C^\infty(T^*M)$, and the Koszul differential ∂ for the zero level of the momentum map (in this case, the ideal generated by the linear hamiltonians $\{h_{\rho(X)} | X \in \mathfrak{g}\}$) [27]. On generators, we have

$$df(Y) = \{h_{\rho(Y)}, \pi^* f\} = \pi^* \rho(Y) f = \pi^* d_A f(Y)$$

for $f \in C^\infty(M)$ and for all $Y \in \mathfrak{g}$. Hence, $df = \{\mu, \pi^* f\}$;

$$d\xi(X, Y) = -\xi([X, Y]) = \pi^* d_A \xi(X, Y)$$

for $\xi \in \mathfrak{g}^*$ (a constant section of A^*). Hence, $d\xi = \{\theta, \pi^* \xi\}$;

$$dh_v(Y) = \{h_{\rho(Y)}, h_V\} = h_{[\rho(Y), v]}$$

for any vector field v on M . Hence, $dh_v = h_{[d_A, v]} = \{\mu, h_v\}$; and finally, for $X \in \mathfrak{g}$,

$$\delta X(Y) = ad_Y X = -[X, Y]; \quad \partial X = h_{\rho(X)}$$

so

$$dX = h_{\rho(X) + ad_X^*} = h_{[d_A, i_X]} = \{\mu, h_{i_X}\}$$

where ad_X^* is viewed as a vector field on $\Pi\mathfrak{g}$. Thus the BRST differential d coincides with our differential $\{\mu, \cdot\}$.

Example 3.6.10. Let \mathfrak{g} be a Lie algebra, then \mathfrak{g}^* is a Poisson manifold, with the canonical linear Poisson structure. Consider the corresponding Lie bialgebroid $A = T\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}^*$, $A^* = T^*\mathfrak{g}^* \simeq \mathfrak{g}^* \times \mathfrak{g}$. Then $C^\infty(T^*\Pi A) = C^\infty(\mathfrak{g}^* \oplus \mathfrak{g}) \otimes \wedge(\mathfrak{g}^* \oplus \mathfrak{g})$. A choice of a basis $\{e_a\}$ of \mathfrak{g} and a dual basis $\{e^a\}$ of \mathfrak{g}^* gives rise to coordinates (u_a, θ_a) on ΠA and (u_a, ξ^a) on ΠA^* . Then the differentials are the deRham differential

$$d = \theta_a \frac{\partial}{\partial u_a}$$

on ΠA , and the Poisson differential

$$d_\pi = u_a C_{bc}^a \xi^b \frac{\partial}{\partial u_c} - \frac{1}{2} C_{ab}^c \xi^a \xi^b \frac{\partial}{\partial \xi^c}$$

where C_{ab}^c are the structure constants of \mathfrak{g} . Thus,

$$\theta = u_a C_{bc}^a \theta_*^b u_*^c - \frac{1}{2} C_{ab}^c \theta_*^a \theta_*^b \theta_c + \theta_a u_*^a$$

on $T^*\Pi A$ and

$$\begin{aligned} D = \{\theta, \cdot\} &= (\theta_c + u_a C_{bc}^a \theta_*^b) \frac{\partial}{\partial u_c} + (\theta_*^b C_{bc}^a \theta_a + u_a C_{cb}^a u_*^b) \frac{\partial}{\partial \theta_c} + \\ &+ C_{ab}^c u_*^a \theta_*^b \frac{\partial}{\partial u_*^c} + (u_*^c - \frac{1}{2} C_{ab}^c \theta_*^a \theta_*^b) \frac{\partial}{\partial \theta_*^c} \end{aligned}$$

Notice that the fibre over the origin, given by the equations $u_a = \theta_a = 0$, is a Lagrangian submanifold F stable under D . The restriction of D to F is

$$D = C_{ab}^c u_*^a \theta_*^b \frac{\partial}{\partial u_*^c} + (u_*^c - \frac{1}{2} C_{ab}^c \theta_*^a \theta_*^b) \frac{\partial}{\partial \theta_*^c}$$

The algebra of polynomial functions on F , isomorphic to $S\mathfrak{g}^* \otimes \wedge\mathfrak{g}^* = \mathbb{R}[u_*^a, \theta_*^a]$, is known as the *Weil algebra* $W(\mathfrak{g})$, while the restricted differential D above is the *Weil differential*. This is Weil's deRham model for the universal classifying space BG , at least when the group G (whose Lie algebra is \mathfrak{g}) is compact [4]. Its appearance in this context is a mystery to us. Notice, however, that our κ -grading, $\kappa(u_*^a) = 2$, $\kappa(\theta_*^a) = 1$, is consistent with the grading in the Weil algebra. Notice also that $T^*\Pi A \simeq T^*F$ by a Legendre transform; after this identification, the “full” D is just the hamiltonian lift of the Weil differential.

3.7 The Courant Algebroid

The Courant algebroid constructed in [31] as the double of a Lie bialgebroid (A, A^*) (see Example 2.6.7) can be recovered from the supermanifold double $(T^*\Pi A, D)$ via the derived bracket construction. We shall view sections of $A \oplus A^*$ and functions on M as functions on $\Pi(A \oplus A^*)$ and use the projection p (see (3.14)).

Theorem 3.7.1. *Let $(E = A \oplus A^*, \langle \cdot, \cdot \rangle, \circ, \rho)$ be as in Example 2.6.7. Then, for any $e_1, e_2 \in \Gamma(E)$, $f \in C^\infty(M)$ we have*

$$1. \quad p^* \langle e_1, e_2 \rangle = \{p^* e_1, p^* e_2\}$$

$$2. \quad p^* \mathcal{D}f = Dp^* f = \{\theta, p^* f\}$$

$$3. \quad p^*(e_1 \circ e_2) = p^* e_1 \circ_D p^* e_2$$

Proof. (1) is just a restatement of Corollary 3.4.4; (2) follows by computation:

$$\begin{aligned} p^* \mathcal{D}f &= p^*(d_A f + d_{A^*} f) = \pi^* d_A f + L^* \bar{\pi}^* d_{A^*} f = \\ &= \{\mu, \pi^* f\} + L^* \{\gamma, \bar{\pi}^* f\} = \{\mu + L^* \gamma, \pi^* f\} = Dp^* f \end{aligned}$$

(3) takes a bit more work. We have

$$\begin{aligned} p^* X \circ_D p^* Y &= \{\{\mu + L^* \gamma, h_{i_X}\}, h_{i_Y}\} = \\ &= \{h_{[d_A, i_X]}, h_{i_Y}\} + L^* \{\{\gamma, \bar{\pi}^* X\}, \bar{\pi}^* Y\} = \\ &= h_{[L_X^A, i_Y]} + L^* \{\bar{\pi}^* d_{A^*} X, \bar{\pi}^* Y\} = \\ &= h_{i_{[X, Y]_A}} = p^*[X, Y]_A \end{aligned}$$

for $X, Y \in \Gamma(A)$;

$$\begin{aligned} p^* \xi \circ_D p^* \eta &= \{\{\mu + L^* \gamma, \pi^* \xi\}, \pi^* \eta\} = \\ &= \{\{\mu, \pi^* \xi\}, \pi^* \eta\} + L^* \{\{\gamma, h_{i_\xi}\}, h_{i_\eta}\} = \\ &= L^* \{h_{[d_{A^*}, i_\xi]}, h_{i_\eta}\} = L^* h_{[L_\xi^{A^*}, i_\eta]} = \\ &= L^* h_{i_{[\xi, \eta]_{A^*}}} = \pi^* [\xi, \eta]_{A^*} = p^* [\xi, \eta]_{A^*} \end{aligned}$$

for $\xi, \eta \in \Gamma(A^*)$;

$$\begin{aligned}
p^*X \circ_D p^*\eta &= \{\{\mu + L^*\gamma, h_{i_X}\}, \pi^*\eta\} = \\
&= \{h_{[d_A, i_X]}, \pi^*\eta\} + L^*\{\{\gamma, \bar{\pi}^*X\}, h_{i_\eta}\} = \\
&= \{h_{L_X^A}, \pi^*\eta\} + L^*\{\bar{\pi}^*d_{A^*}X, h_{i_\eta}\} = \\
&= \pi^*L_X^A\eta - L^*\bar{\pi}^*i_\eta d_{A^*}X = \\
&= p^*(L_X^A\eta - i_\eta d_{A^*}X)
\end{aligned}$$

for $X \in \Gamma(A)$, $\eta \in \Gamma(A^*)$; and finally,

$$\begin{aligned}
p^*\xi \circ_D p^*Y &= \{\{\mu + L^*\gamma, \pi^*\xi\}, h_{i_Y}\} = \\
&= \{\{\mu, \pi^*\xi\}, h_{i_Y}\} + L^*\{h_{[d_{A^*}, i_\xi]}, \bar{\pi}^*Y\} = \\
&= \{\pi^*d_A\xi, h_{i_Y}\} + L^*\{h_{L_\xi^{A^*}}, \bar{\pi}^*Y\} = \\
&= -\pi^*i_Y d_A\xi + L^*\bar{\pi}^*L_\xi^{A^*}Y = \\
&= p^*(-i_Y d_A\xi + L_\xi^{A^*}Y)
\end{aligned}$$

This proves (3). We have made extensive use of the commutation relations (2.4) and Lemmas 3.3.1 and 3.4.3. \square

Remark 3.7.2. The Theorem above is true regardless of whether (A, A^*) is a Lie bialgebroid, i.e. whether (3.17) holds; however, if it is the case, we can use the differential Lie superalgebra $(C^\infty(T^*\Pi A), \{\cdot, \cdot\}, D)$ and its derived bracket to prove that $(A \oplus A^*, \langle \cdot, \cdot \rangle, \circ, \rho)$ actually is a Courant algebroid, thus recovering the doubling theorem of Liu, Weinstein and Xu:

Theorem 3.7.3. *If (A, A^*) is a Lie bialgebroid, then $(A \oplus A^*, \langle \cdot, \cdot \rangle, \circ, \rho)$ is a Courant algebroid.*

Proof. We need to verify properties 1-5 of Definition 2.6.1. Since p is a Poisson map, we can embed sections of $A \oplus A^*$ and functions on M into $C^\infty(T^*\Pi A)$ using p^* as above and carry out all the computations up in $C^\infty(T^*\Pi A)$. We shall identify $e_i \in \Gamma(A \oplus A^*)$ and $f \in C^\infty(M)$ with their images under p^* .

Now, it follows that properties 1 (the Leibniz-Jacobi identity) and 4 (about the symmetric part) are just consequences of the properties of the derived bracket on a differential Lie superalgebra (Lemma 3.5.1). On the other hand, Property 3,

$$e_1 \circ f e_2 = f(e_1 \circ e_2) + (\rho(e_1)f)e_2$$

translates, by Theorem 3.7.1, into

$$\{\{\theta, e_1\}, f e_2\} = \{\{\theta, e_1\}, f\} e_2 + f \{\{\theta, e_1\}, e_2\},$$

but this is obvious. Property 5,

$$\rho(e) \langle e_1, e_2 \rangle = \langle e \circ e_1, e_2 \rangle + \langle e_1, e \circ e_2 \rangle$$

translates into

$$\{e, \{\theta, \{e_1, e_2\}\}\} = \{\{\{\theta, e\}, e_1\}, e_2\} + \{e_1, \{\{\theta, e\}, e_2\}\}$$

However, by the Jacobi identity for $\{\cdot, \cdot\}$,

$$\begin{aligned} \{e, \{\theta, \{e_1, e_2\}\}\} &= \{\{e, \theta\}, \{e_1, e_2\}\} - \{\theta, \{e, \{e_1, e_2\}\}\} = \\ &= \{\{\{\theta, e\}, e_1\}, e_2\} + \{e_1, \{\{\theta, e\}, e_2\}\} \end{aligned}$$

since $\{e, \{e_1, e_2\}\} = 0$ because $\{e_1, e_2\} \in C^\infty(M)$ is a Casimir function for $\Pi(A \oplus A^*)$. Finally, property 2,

$$\rho(e_1 \circ e_2) = [\rho(e_1), \rho(e_2)]$$

when both sides are applied to an arbitrary $f \in C^\infty(M)$ translates into

$$\{\{\{\theta, e_1\}, e_2\}, \{\theta, f\}\} = \{e_1, \{\theta, \{e_2, \{\theta, f\}\}\}\} - \{e_2, \{\theta, \{e_1, \{\theta, f\}\}\}\}$$

for all $f \in C^\infty(M)$. Using Jacobi again, we have

$$\begin{aligned} -\{\{e_2, \{e_1, \theta\}\}, \{\theta, f\}\} &= -\{e_2, \{\{e_1, \theta\}, \{\theta, f\}\}\} + \{\{e_1, \theta\}, \{e_2, \{\theta, f\}\}\} = \\ &= -\{e_2, \{\theta, \{e_1, \{\theta, f\}\}\}\} - \{e_2, \{e_1, \{\theta, \{\theta, f\}\}\}\} + \\ &+ \{\theta, \{e_1, \{e_2, \{\theta, f\}\}\}\} + \{\{e_1, \{\theta, \{e_2, \{\theta, f\}\}\}\}\} = \\ &= \{\{e_1, \{\theta, \{e_2, \{\theta, f\}\}\}\}\} - \{e_2, \{\theta, \{e_1, \{\theta, f\}\}\}\} \end{aligned}$$

since $\{\theta, \{\theta, f\}\} = 0$ by (3.17), while $\{e_1, \{e_2, \{\theta, f\}\}\} = 0$ since $\{e_2, \{\theta, f\}\} \in C^\infty(M)$ is a Casimir function on $\Pi(A \oplus A^*)$.

Thus all of the properties of a Courant algebroid are verified. Notice that (3.17) was only needed to derive properties 1 and 2. \square

3.8 Quasi-bialgebroids

The hamiltonian θ we constructed above was a sum of two terms, μ of bidegree $(1, 2)$, and $L^*\gamma$ of bidegree $(2, 1)$, so it has total degree $\kappa(\theta) = 3$. There is nothing to prevent us from adding a ϕ of bidegree $(0, 3)$ and/or a ψ of bidegree $(3, 0)$ to θ , and require that $\{\theta, \theta\} = 0$.

Definition 3.8.1. A *proto-bialgebroid* is the supermanifold $T^*\Pi A$ together with a function θ such that $\kappa(\theta) = 3$ and $\{\theta, \theta\} = 0$.

Thus, a proto-bialgebroid structure consists of a vector field d_A on ΠA , a vector field d_{A^*} on ΠA^* , and two functions $\phi \in \Gamma(\bigwedge^3 A^*) \subset C^\infty(\Pi A)$ and $\psi \in \Gamma(\bigwedge^3 A) \subset C^\infty(\Pi A^*)$. Then $\theta = \mu + L^*\gamma + \pi^*\phi + L^*\bar{\pi}^*\psi$, and the equation $\{\theta, \theta\} = 0$ splits according to the bigrading into the following five equations:

$$\begin{aligned} \frac{1}{2}\{\mu, \mu\} + \{L^*\gamma, \pi^*\phi\} &= 0 \\ \{\mu, L^*\gamma\} + \{\pi^*\phi, L^*\bar{\pi}^*\psi\} &= 0 \\ \frac{1}{2}L^*\{\gamma, \gamma\} + \{\mu, L^*\bar{\pi}^*\psi\} &= 0 \\ \{\mu, \pi^*\phi\} = \{\gamma, \bar{\pi}^*\psi\} &= 0 \end{aligned} \tag{3.18}$$

In particular, $d_A\phi = d_{A^*}\psi = 0$ and the Schouten brackets $[\cdot, \cdot]_A$ and $[\cdot, \cdot]_{A^*}$ are defined, but neither d_A nor d_{A^*} square to zero, nor is d_A a derivation of $[\cdot, \cdot]_{A^*}$. The defects in all cases are determined by the above relations.

Nevertheless, since the Poisson bracket on $T^*\Pi A$ has total degree -2 , $p^*\Gamma(A \oplus A^*)$ will be closed under both the Poisson bracket and the derived bracket for any proto-bialgebroid, since elements of $p^*\Gamma(A \oplus A^*)$ have total degree 1. Thus, a slight modification of Theorem 3.7.1 to include ϕ and ψ , and repeating the argument of Theorem 3.7.3 yields

Theorem 3.8.2. Any proto-bialgebroid structure on $T^*\Pi A$ induces a Courant algebroid structure on the bundle $A \oplus A^*$ given by

$$\begin{aligned} \langle X + \xi, Y + \eta \rangle &= \eta(X) + \xi(Y) \\ (X + \xi) \circ (Y + \eta) &= ([X, Y]_A + L_\xi^{A^*}Y - i_\eta d_{A^*}X - \psi(\xi, \eta)) + \\ &\quad + ([\xi, \eta]_{A^*} + L_X^A\eta - i_Y d_A\xi - \phi(X, Y)) \\ \mathcal{D}f &= d_A f + d_{A^*} f \end{aligned}$$

where ϕ is viewed as a bundle map $\phi : \bigwedge^2 A \rightarrow A^*$ and likewise, $\psi : \bigwedge^2 A^* \rightarrow A$.

We will consider the special case where either ϕ or ψ is zero, say, $\psi = 0$. The equations (3.18) reduce to

$$\begin{aligned}\frac{1}{2}\{\mu, \mu\} + \{L^*\gamma, \phi\} &= 0 \\ \{\gamma, \gamma\} &= 0 \\ \{\mu, L^*\gamma\} &= 0 \\ \{\mu, \phi\} &= 0\end{aligned}$$

Deciphering these equations we arrive at

Definition 3.8.3. A *quasi-Lie bialgebroid* structure on (A, A^*) consists of the following data:

- A Lie algebroid structure on A^*
- A bundle map $a : A \rightarrow TM$
- A skew-symmetric operation $[\cdot, \cdot]_A$ on $\Gamma(A)$
- An element $\phi \in \Gamma(\bigwedge^3 A^*)$

satisfying the following properties:

1. For all $X, Y \in \Gamma(A)$, $f \in C^\infty(M)$,

$$[X, fY]_A = f[X, Y]_A + (a(X)f)Y$$

2. For all $X, Y \in \Gamma(A)$,

$$a([X, Y]_A) = [a(X), a(Y)] + a_*\phi(X, Y)$$

where a_* is the anchor of the Lie algebroid A^* and $\phi(X, Y) = i_{X \wedge Y} \phi \in \Gamma(A^*)$.

3. For all $X, Y, Z \in \Gamma(A)$,

$$\begin{aligned} & [[X, Y]_A, Z]_A + [[Y, Z]_A, X]_A + [[Z, X]_A, Y]_A = d_{A^*}\phi(X, Y, Z) + \\ & + \phi(d_{A^*}X, Y, Z) - \phi(X, d_{A^*}Y, Z) + \phi(X, Y, d_{A^*}Z) \end{aligned}$$

where d_{A^*} is the differential on $\Gamma(\bigwedge A)$ coming from the Lie algebroid structure on A^* , and ϕ is viewed as a bundle map $\bigwedge^4 A \rightarrow A$.

4. $d_A\phi = 0$ where d_A is the differential on $\Gamma(\wedge A^*)$ coming from the structure $(a, [\cdot, \cdot]_A)$ on A .

Notice that this is completely analogous to Drinfeld's *quasi-Lie bialgebras* [22]. Property 3 above is to be interpreted as a homotopy Jacobi identity for $[\cdot, \cdot]_A$.

Corollary 3.8.4. *A quasi-Lie bialgebroid structure on (A, A^*) gives rise to a Courant algebroid structure on $A \oplus A^*$.*

Finally, we will look at an important special case of this, *exact Courant algebroids*, which were recently studied and classified by Ševera [43]. A Courant algebroid E is called *exact* if the sequence

$$0 \longrightarrow T^*M \xrightarrow{\rho^*} E \xrightarrow{\rho} TM \longrightarrow 0$$

is exact, where the *co-anchor* $\rho^* : T^*M \rightarrow E$ is given by

$$\langle \rho^*\xi, e \rangle = \xi(\rho(e))$$

for all $\xi \in T^*M$, $e \in E$ (By Property 4 of Definition 2.6.1, $\rho \circ \rho^* = 0$ in any Courant algebroid). Then the image of T^*M is a Dirac subbundle, and the restriction of \circ to its sections is identically zero, by Lemma 2.6.2. One then chooses a “connection” on E , i.e. an isotropic splitting $\sigma : TM \rightarrow E$ of the above exact sequence. This is not a problem: once we have one isotropic subbundle T^*M , transversal isotropic subbundles are sections of a bundle over M whose fiber is an open cell in the Grassmanian of isotropic subspaces of half dimension in a pseudo-Euclidean space of signature zero; the fiber is contractible (it is diffeomorphic to the linear space of skew-symmetric matrices), so sections always exist. The connection σ identifies the pseudo-Euclidean vector bundle E with $TM \oplus T^*M$ with the canonical inner product. To compute the Courant bracket on E in this identification, one looks at the difference

$$\sigma(X) \circ \sigma(Y) - \sigma([X, Y]) = \rho^*\phi(X, Y)$$

where X, Y are vector fields; this holds because σ is a splitting; moreover, using the properties of a Courant algebroid, one immediately deduces that ϕ is $C^\infty(M)$ -linear and completely skew-symmetric, i.e. comes from a 3-form $\phi \in \Omega^3(M)$, which it is appropriate to call the

“curvature” of σ . From the Leibniz-Jacobi identity for \circ (Property 1 of Definition 2.6.1) one deduces the “Bianchi identity”

$$d\phi = 0$$

The Courant bracket becomes

$$(X + \xi) \circ (Y + \eta) = [X, Y] + L_X\eta - i_Y d\xi + \phi(X, Y)$$

where $X, Y \in \mathfrak{X}(M)$, $\xi, \eta \in \Omega^1(M)$ and $\phi(X, Y) = i_{X \wedge Y} \phi \in \Omega^1(M)$. Thus, any exact Courant algebroid comes from a quasi-Lie bialgebroid which is in fact the standard Lie bialgebroid (TM, T^*M) with an additional piece of data, the closed 3-form ϕ which twists the standard Courant bracket (2.20) on $TM \oplus T^*M$.

Once a connection σ is chosen, any other one, σ' , differs from σ by the graph of a 2-form ω ; its curvature ϕ' is related to ϕ simply by

$$\phi' = \phi + d\omega$$

Therefore, the cohomology class $c = [\phi] \in H^3(M, \mathbb{R})$ is independent of the choice of σ and completely determines the Courant algebroid structure on E . It is thus appropriate to call $c = c(E)$ the *characteristic class* of E . This classification of exact Courant algebroids is due to P. Ševera [43].

Example 3.8.5. Let G be a compact semisimple Lie group, with Lie algebra \mathfrak{g} and the Killing form (\cdot, \cdot) . Then Cartan’s *structure tensor*

$$\phi(X, Y, Z) = \frac{1}{12}([X, Y], Z)$$

is the canonical bi-invariant 3-form on G that gives a non-trivial twisting of the standard Courant algebroid structure on $TG \oplus T^*G$. This Courant algebroid plays a role in the recently developed theory of group-valued momentum maps [1] [2].

It is also well-known that $H^3(G, \mathbb{R})$, which is generated by $[\phi]$, classifies Kac-Moody central extensions of the loop algebra $L\mathfrak{g}$ [10]. It is a very interesting question what the above Courant algebroid has to do with affine Kac-Moody algebras.

Remark 3.8.6. As a final remark, we note that Ševera’s classification of exact Courant algebroids is completely analogous to the well-known classification of central extensions

$$0 \longrightarrow \mathbb{R} \longrightarrow E \xrightarrow{a} TM \longrightarrow 0$$

of vector fields by functions. The exact sequence above is known as an *Atiyah sequence*. E is then a Lie algebroid, and the kernel of the anchor a is the trivial one-dimensional vector bundle. Such Lie algebroids are classified by $H^2(M, \mathbb{R})$; if the characteristic class $c(E)$ is integral, the Atiyah sequence integrates to a principal $U(1)$ -bundle $P \rightarrow M$ and $c(E) = c_1(P)$ is the first Chern class of P . We thus recover the classification of complex line bundles on M .

Now, the meaning of the integrality of the characteristic class of an exact Courant algebroid is still unknown. It is a very interesting question related to the existence of a “global” object for a Courant algebroid, like the principal $U(1)$ -bundle above, or its gauge groupoid. This was posed as an open problem in [31], and there is as yet no solution. Ševera [43] suggests that the answer should come from *Dixmier-Douady gerbes*, but no global object for gerbes is known, either, nor is there a direct correspondence between Courant algebroids and gerbes. Investigating these and related questions are a logical continuation of this work.

Chapter 4

Poisson Cohomology of $SU(2)$ -covariant Poisson structures on S^2

In this chapter we shall compute the Poisson cohomology of the one-parameter family of $SU(2)$ -covariant Poisson structures on the homogeneous space $S^2 = \mathbb{C}P^1 = SU(2)/U(1)$, where $SU(2)$ is endowed with its standard Poisson-Lie group structure, thus extending the result of Ginzburg [17] on the Bruhat-Poisson structure which is a member of this family. As a corollary of our computation, we deduce that these structures are nontrivial deformations of each other in the direction of the standard rotation-invariant symplectic structure on S^2 ; another corollary is that these structures do not admit rescaling.

4.1 Poisson-Lie groups and Poisson actions

Here we briefly recall some basic notions of the theory of Poisson-Lie groups that we will need. For more details the interested reader should consult [10], [21], or [33].

Notation 4.1.1. Let a Lie group G act on a manifold P . Then each $g \in G$ gives rise to a map $P \rightarrow P$ given by $p \mapsto gp$. We shall denote this map as well as its derivatives and their tensor products by the same letter g where it does not cause confusion. Likewise, every $p \in P$ induces a map $G \rightarrow P$ by $g \mapsto gp$ which, along with its derivatives, we shall denote by p written on the right of the argument. This will make our notation a lot less cumbersome.

Definition 4.1.2. A Poisson structure π on a Lie group G is called *multiplicative* if the group multiplication

$$m : G \times G \longrightarrow G$$

is a Poisson map, where $G \times G$ is equipped with the product Poisson structure. The pair (G, π) is then called a *Poisson-Lie group*.

One checks that the multiplicativity condition is equivalent to the identity

$$\pi(gh) = g\pi(h) + \pi(g)h \quad \forall g, h \in G \quad (4.1)$$

In particular, one has $\pi(e) = 0$, so the linearization (intrinsic derivative) of π at e gives a well-defined *cobacket* $\sigma : \mathfrak{g} \rightarrow \mathfrak{g} \wedge \mathfrak{g}$ by

$$\sigma(X) = (L_{X_l}\pi)(e) = \left. \frac{d}{dt} \right|_{t=0} \pi(\exp(tX)) \exp(-tX),$$

where X_l denotes the left-invariant vector field corresponding to $X \in \mathfrak{g}$. The multiplicativity of π (4.1) then implies the cocycle property of σ :

$$\sigma([X, Y]) = [X, \sigma(Y)] - [Y, \sigma(X)]$$

On the other hand, the Jacobi identity for π implies that the adjoint of σ , $\sigma^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \rightarrow \mathfrak{g}^*$ also satisfies Jacobi, i.e. defines a Lie bracket on \mathfrak{g}^* . Thus, $(\mathfrak{g}, [\cdot, \cdot], \sigma)$ is a Lie bialgebra (see Section 2.1) called the *tangent Lie bialgebra* of the Poisson-Lie group G . It can be shown [33] that if G is connected, π is uniquely determined by σ .

It may happen that the cocycle σ is a coboundary, that is, there exists an $\mathbf{r} \in \mathfrak{g} \wedge \mathfrak{g}$ such that $\sigma(X) = -[X, \mathbf{r}]$ (always the case if \mathfrak{g} is semisimple). Such an \mathbf{r} is called a *classical r -matrix*. The Jacobi identity for σ^* is equivalent to the condition on \mathbf{r} that $[\mathbf{r}, \mathbf{r}] \in \bigwedge^3 \mathfrak{g}$ be *ad*-invariant (the so-called Modified Classical Yang-Baxter Equation). Here $[\cdot, \cdot]$ is the algebraic Schouten bracket of the Lie algebra \mathfrak{g} (Example 3.5.4). The multiplicative Poisson structure π is given in terms of \mathbf{r} by

$$\pi(g) = \mathbf{r}g - g\mathbf{r}, \quad (4.2)$$

and the corresponding Poisson bracket on G is called the *Sklyanin bracket*.

Definition 4.1.3. Let a Poisson-Lie group (G, π_G) act on a manifold P . We say that a Poisson structure π_P on P is *G-covariant* if the action map

$$\rho : G \times P \longrightarrow P$$

is Poisson, where $G \times P$ is equipped with the product Poisson structure. The action ρ is then called a *Poisson action*. If ρ is transitive, (P, π_P) is called a *Poisson homogeneous space*.

The covariance condition is equivalent to the identity

$$\pi_P(gp) = \pi_G(g)p + g\pi_P(p) \quad \forall g \in G, p \in P \quad (4.3)$$

Note that G does *not* act by Poisson transformations unless $\pi_G = 0$.

Fact 4.1.4. [21] *If (G, π_G) is a Poisson-Lie group, $H \subset G$ a Poisson (or even coisotropic) subgroup, then there is a unique Poisson structure π_P on $P = G/H$ making the canonical projection a Poisson map. Moreover, π_P is G-covariant.*

So if H is coisotropic, G/H is always a Poisson homogeneous space; however, the projection of π_G is in general not the only G -covariant Poisson structure on G/H : adding any G -invariant bivector field will give another one provided that the sum satisfies the Jacobi identity.

4.2 Description of the Poisson structures

4.2.1 The classical r-matrix and the standard Poisson-Lie structure on $SU(2)$.

The constructions below can be carried out for any compact semisimple Lie group, but we will only consider $SU(2)$.

Recall that the Lie algebra $\mathfrak{su}(2)$ of 2×2 skew-hermitian traceless matrices has a basis

$$e_1 = \frac{1}{2} \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad e_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad e_3 = \frac{1}{2} \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

with the commutation relations $[e_\alpha, e_\beta] = \epsilon_{\alpha\beta\gamma} e_\gamma$, where $\epsilon_{\alpha\beta\gamma}$ is the completely skew-symmetric symbol. The span of e_1 is the Cartan subalgebra of $\mathfrak{a} \in \mathfrak{su}(2)$. Recall also that

$$SU(2) = \left\{ U = \begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \middle| u, v \in \mathbb{C}, \det U = u\bar{u} + v\bar{v} = 1 \right\}$$

identifies $SU(2)$ with the unit sphere in \mathbb{C}^2 . The *standard r-matrix* $\mathbf{r} = e_2 \wedge e_3 \in \mathfrak{su}(2) \wedge \mathfrak{su}(2)$ defines a multiplicative Poisson structure on $SU(2)$ by

$$\pi_{SU(2)}(U) = \mathbf{r}U - U\mathbf{r} \quad (4.4)$$

In coordinates,

$$\begin{aligned} \pi \left(\begin{pmatrix} u & -\bar{v} \\ v & \bar{u} \end{pmatrix} \right) &= \frac{1}{4} \left(\begin{pmatrix} v & \bar{u} \\ -u & \bar{v} \end{pmatrix} \wedge \begin{pmatrix} iv & i\bar{u} \\ iu & -i\bar{v} \end{pmatrix} - \begin{pmatrix} \bar{v} & u \\ -\bar{u} & v \end{pmatrix} \wedge \begin{pmatrix} -i\bar{v} & iu \\ i\bar{u} & iv \end{pmatrix} \right) = \\ &= -iv\bar{v} \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{u}} + \frac{1}{2} \left(iuv \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v} + \overline{iuv \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial v}} \right) + \frac{1}{2} \left(iu\bar{v} \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{v}} + \overline{iu\bar{v} \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial \bar{v}}} \right) \end{aligned} \quad (4.5)$$

The Poisson brackets are

$$\{u, \bar{u}\} = -iv\bar{v}, \quad \{u, v\} = \frac{1}{2}iuv, \quad \{u, \bar{v}\} = \frac{1}{2}iu\bar{v}, \quad \{v, \bar{v}\} = 0$$

It is easy to see that these formulas in fact define a smooth real Poisson structure on all of \mathbb{C}^2 that restricts to the unit sphere.

4.2.2 The Bruhat-Poisson structure on \mathbb{CP}^1 .

The r-matrix is invariant under the action of the Cartan subalgebra \mathfrak{a} , since

$$[e_1, \mathbf{r}] = [e_1, e_2 \wedge e_3] = [e_1, e_2] \wedge e_3 - e_2 \wedge [e_1, e_3] = e_3 \wedge e_3 + e_2 \wedge e_2 = 0$$

Hence, the Poisson tensor (4.4) vanishes on the maximal torus (the diagonal subgroup) $A = U(1) \subset SU(2)$. In particular, $U(1)$ is a Poisson subgroup, and hence $\pi_{SU(2)}$ descends to the quotient $SU(2)/U(1) = S^3/S^1 = (\mathbb{C}^2 \setminus 0)/\mathbb{C}^\times = \mathbb{CP}^1 = S^2$. The resulting Poisson structure π_1 on \mathbb{CP}^1 is called the *Bruhat-Poisson structure* because its symplectic leaves coincide with the Bruhat cells in \mathbb{CP}^1 [33]: the base point where π_1 vanishes, and the complementary open

cell where π_1 is invertible. It is $SU(2)$ -covariant since $\pi_{SU(2)}$ is multiplicative. It is an easy calculation to deduce from (4.5) that in the inhomogeneous coordinate chart $w = v/u$ covering the base point π_1 is given by

$$\pi_1 = -iw\bar{w}(1 + w\bar{w})\frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}}$$

In particular, it has a quadratic singularity at $w = 0$. The other inhomogeneous chart $z = u/v = 1/w$ gives coordinates on the open symplectic leaf, in which

$$\pi_1 = -i(1 + z\bar{z})\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \bar{z}}$$

The corresponding symplectic 2-form is

$$\omega_1 = \frac{idz \wedge d\bar{z}}{1 + z\bar{z}}$$

Notice that this symplectic leaf has infinite volume.

4.2.3 The other $SU(2)$ -covariant Poisson structures on S^2 .

The difference between any two $SU(2)$ - covariant Poisson structures on \mathbb{CP}^1 is an $SU(2)$ - invariant bivector field (by (4.3)) which is Poisson because in two dimensions, any bivector field is. Thus, any covariant structure is obtained by adding an invariant structure to the Bruhat structure π_1 . To see what these structures look like, it is convenient to embed the Riemann sphere \mathbb{CP}^1 as the unit sphere $S^2 \subset \mathbb{R}^3$ by the (inverse of) the stereographic projection. The coordinate transformations are given by

$$\begin{aligned} x_1 &= \frac{2x}{1+x^2+y^2} & x &= \frac{x_1}{1-x_3} \\ x_2 &= \frac{2y}{1+x^2+y^2} & y &= \frac{x_2}{1-x_3} \\ x_3 &= \frac{x^2+y^2-1}{1+x^2+y^2} & x^2+y^2 &= \frac{1+x_3}{1-x_3} \end{aligned}$$

where $z = x + iy$. We shall identify \mathbb{R}^3 with $\mathfrak{su}(2)^*$, with the coadjoint action of $SU(2)$ by rotations. Then the linear Poisson structure on $\mathbb{R}^3 = \mathfrak{su}(2)^*$ is given by

$$-\pi = x_1 \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_3} + x_2 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_1} + x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_2}$$

whose restriction to the unit sphere (a coadjoint orbit), also denoted by $-\pi$, is $SU(2)$ - invariant and symplectic. Moreover, up to a constant multiple, π is the only rotation-invariant Poisson structure on S^2 : any other invariant structure is of the form $\pi' = f\pi$ for

some function f , but since both π and π' are invariant, so is f , hence f is a constant. It follows that there is a one-parameter family of $SU(2)$ - covariant Poisson structures of the form $\pi' = \pi_1 + \alpha\pi$, $\alpha \in \mathbb{R}$; since $\pi_1 = (1 - x_3)\pi$ (straightforward calculation), all $SU(2)$ - covariant structures are of the form

$$\pi_c = \pi_1 + (c - 1)\pi = (c - x_3)\pi, \quad c \in \mathbb{R}$$

It follows that π_c is symplectic for $|c| > 1$, Bruhat for $c = \pm 1$, while for $|c| < 1$ π_c vanishes on the circle $\{x_3 = c\}$ and is nonsingular elsewhere; π_c thus has two open symplectic leaves (hemispheres) and a “necklace” of zero-dimensional symplectic leaves along the circle. It is these “necklace” structures whose Poisson cohomology we shall compute. Notice that π_c and π_{-c} are isomorphic as Poisson manifolds via $x_3 \mapsto -x_3$.

In the original $\{w, \bar{w}\}$ - coordinates we have

$$\pi = -\frac{i}{2}(1 + w\bar{w})^2 \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}} = \frac{1}{4}(1 + x^2 + y^2)^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}, \quad (4.6)$$

$$\begin{aligned} \pi_c &= \pi_1 + (c - 1)\pi = -\frac{i}{2}(1 + w\bar{w})((c + 1)w\bar{w} + c - 1) \frac{\partial}{\partial w} \wedge \frac{\partial}{\partial \bar{w}} = \\ &= \frac{1}{4}(1 + x^2 + y^2)((c + 1)(x^2 + y^2) + c - 1) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \end{aligned} \quad (4.7)$$

where $w = x + iy$.

4.2.4 Symplectic areas and modular vector fields.

Before we proceed to cohomology computations, we shall compute some invariants of the structures π_c . For $|c| > 1$ π_c is symplectic, and the only invariant is the symplectic area. For the other values of c , the areas of the open symplectic leaves are easily seen to be infinite; instead, we will compute the modular vector field of π_c with respect to the standard rotation-invariant volume form ω on S^2 (the inverse of π). By elementary calculations we obtain the following

Lemma 4.2.1. (1) If $|c| > 1$, the symplectic volume of (S^2, π_c) is given by

$$V(c) = 2\pi \ln \frac{c + 1}{c - 1}$$

(2) For all values of c the modular vector field with respect to ω is

$$\Delta_\omega = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}$$

Corollary 4.2.2. *If $|c|, |c'| > 1$, π_c and $\pi_{c'}$ are not isomorphic unless $|c| = |c'|$.*

Corollary 4.2.3. *If $|c| < 1$, the modular class of π_c is nonzero.*

Proof. The modular vector field Δ_ω rotates the necklace, hence cannot be Hamiltonian. \square

In fact, the modular class of the Bruhat-Poisson structures $\pi_{\pm 1}$ is also nonzero [17].

Unfortunately, the modular vector field does not help us distinguish the different “necklace” structures. The restriction of Δ_ω to the necklace is independent of ω since changing ω changes Δ_ω by a Hamiltonian vector field which necessarily vanishes along the necklace, so the period of Δ_ω restricted to the necklace is an invariant, but it has the same value of 2π for all π_c . When we compute the Poisson cohomology of π_c we will see a different way to distinguish them.

4.3 Computation of Poisson cohomology

For $|c| > 1$ π_c is symplectic, so its Poisson cohomology is isomorphic to the deRham cohomology of S^2 ; the Poisson cohomology of the Bruhat-Poisson structure $\pi_{\pm 1}$ was worked out by Ginzburg [17]. Here we shall compute the cohomology of the necklace structures π_c for $|c| < 1$. Our strategy will be similar to Ginzburg’s: first compute the cohomology of the formal neighborhood of the necklace, show that the result is actually valid in a finite small neighborhood and finally, use a Mayer-Vietoris argument to deduce the global result. The validity of the Mayer-Vietoris argument for Poisson cohomology comes from the simple observation that on any Poisson manifold (P, π) the differential d_π is functorial with respect to restrictions to open subsets (i.e. a morphism of the sheaves of smooth multivector fields on P).

It will be convenient to introduce another change of coordinates:

$$s = \frac{x}{\sqrt{1+x^2+y^2}} \quad t = \frac{y}{\sqrt{1+x^2+y^2}}$$

mapping the (x, y) -plane to the open unit disk in the (s, t) -plane. In the new coordinates π_c and π are given by

$$\pi_c = \frac{1}{2}(s^2 + t^2 - \frac{1-c}{2}) \frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t} \quad (4.8)$$

$$\pi = \frac{1}{4} \frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t} \quad (4.9)$$

and the necklace is the circle of radius $R = \sqrt{\frac{1-c}{2}}$. Observe that rescaling $s = \alpha s'$, $t = \alpha t'$ ($\alpha > 0$) takes π_c with necklace radius R to $\pi_{c'}$ with necklace radius $R' = R/\alpha$. But this is only a local isomorphism: it does not extend to all of S^2 since it is not a diffeomorphism of the unit disk. In any case, it shows that all necklace structures are locally isomorphic, so for local computations we may assume that π_c is given in suitable coordinates by

$$\pi_c = \frac{1}{2}(s^2 + t^2 - 1) \frac{\partial}{\partial s} \wedge \frac{\partial}{\partial t}$$

4.3.1 Cohomology of the formal neighborhood of the necklace.

Since π_c is rotation-invariant, we can lift the computations in the formal neighborhood of the unit circle in the (s, t) -plane to its universal cover by introducing “action-angle coordinates” (I, θ) :

$$s = \sqrt{1+I} \cos \theta \quad t = \sqrt{1+I} \sin \theta$$

in which π_c is linear:

$$\pi_c = I \frac{\partial}{\partial I} \wedge \frac{\partial}{\partial \theta}$$

Of course we will have to restrict attention to multivector fields whose coefficients are periodic in θ . It will be convenient to think of multivector fields as functions on the supermanifold with coordinates (I, θ, ξ, η) where $\xi = \partial_I$ and $\eta = \partial_\theta$ are Grassmann (anticommuting) variables. Then $\pi_c = I\xi\eta$ is a function and

$$d_{\pi_c} = [\pi_c, \cdot] = -I\eta \frac{\partial}{\partial I} + I\xi \frac{\partial}{\partial \theta} - \xi\eta \frac{\partial}{\partial \xi}$$

is a (homological) vector field. Since d_{π_c} commutes with rotations, we can split the complex into Fourier modes

$$\mathfrak{X}_n^0 = \{f(I)e^{in\theta}\}; \quad \mathfrak{X}_n^1 = \{(f(I)\xi + g(I)\eta)e^{in\theta}\}; \quad \mathfrak{X}_n^2 = \{h(I)\xi\eta e^{in\theta}\},$$

where $f(I)$, $g(I)$ and $h(I)$ are formal power series in I . It will be convenient to treat the zero and non-zero modes separately; it will turn out that the cohomology is concentrated entirely in the zero mode.

Case 1. The zero mode ($n = 0$) consists of multivector fields independent of θ , so d_{π_c} becomes

$$d_{\pi_c}|_{\mathfrak{X}_0} = -I\eta \frac{\partial}{\partial I} + \eta\xi \frac{\partial}{\partial \xi}$$

which preserves the degree in I so the complex \mathfrak{X}_0 splits further into a direct product of sub-complexes $\mathfrak{X}_{0,m}$, $m \geq 0$ according to the degree:

$$0 \rightarrow \mathfrak{X}_{0,m}^0 \rightarrow \mathfrak{X}_{0,m}^1 \rightarrow \mathfrak{X}_{0,m}^2 \rightarrow 0$$

These complexes are very small ($\mathfrak{X}_{0,m}^0$ and $\mathfrak{X}_{0,m}^2$ are one-dimensional, while $\mathfrak{X}_{0,m}^1$ is two-dimensional) and their cohomology is easy to compute. For $f = cI^m \in \mathfrak{X}_{0,m}^0$, $d_{\pi_c}f = -cmI^m\eta$, while for $X = aI^m\xi + bI^m\eta \in \mathfrak{X}_{0,m}^1$, $d_{\pi_c}X = a(m-1)I^m\xi\eta$. Therefore, it is clear that for $m > 1$ the complex is acyclic. On the other hand, the cohomology of $\mathfrak{X}_{0,0}$ is generated by $1 \in \mathfrak{X}_{0,0}^0$ and $\eta \in \mathfrak{X}_{0,0}^1$, while the cohomology of $\mathfrak{X}_{0,1}$ is generated by $I\xi \in \mathfrak{X}_{0,1}^1$ and $I\xi\eta \in \mathfrak{X}_{0,1}^2$. Putting these together we obtain

$$\begin{aligned} H_0^0 &= \mathbb{R} = \text{span}\{1\} \\ H_0^1 &= \mathbb{R}^2 = \text{span}\{\partial_\theta, I\partial_I\} \\ H_0^2 &= \mathbb{R} = \text{span}\{I\partial_I \wedge \partial_\theta\} \end{aligned} \tag{4.10}$$

Case 2. The non-zero modes ($n \neq 0$). In this case d_{π_c} does not preserve the I -grading so we'll have to consider all power series at once. Let

$$\begin{aligned} f &= (\sum_{m=0}^\infty f_m I^m) e^{in\theta} && \in \mathfrak{X}_n^0 \\ X &= (\sum_{m=0}^\infty a_m I^m) e^{in\theta} \xi + (\sum_{m=0}^\infty b_m I^m) e^{in\theta} \eta && \in \mathfrak{X}_n^1 \\ B &= (\sum_{m=0}^\infty c_m I^m) e^{in\theta} \xi \eta && \in \mathfrak{X}_n^2 \end{aligned}$$

Then

$$\begin{aligned} d_{\pi_c}f &= (\sum_{m=1}^\infty in f_{m-1} I^m) e^{in\theta} \xi + (\sum_{m=1}^\infty m f_m I^m) e^{in\theta} \eta \\ d_{\pi_c}X &= (-a_0 + \sum_{m=1}^\infty ((m-1)a_m + in b_{m-1}) I^m) e^{in\theta} \xi \eta \end{aligned}$$

(and, of course, $d_{\pi_c}B = 0$). We see immediately that $d_{\pi_c}f = 0 \Leftrightarrow f = 0$, hence $H_n^0 = \{0\}$.

Moreover, any B is a coboundary:

$$B = d_{\pi_c} \left(\left(\sum_{m \neq 1}^\infty \frac{c_m}{m-1} I^m \right) e^{in\theta} \xi + \frac{c_1}{in} e^{in\theta} \eta \right)$$

so $H_n^0 = \{0\}$ as well. Now, X is a cocycle if and only if

$$\begin{aligned} a_0 &= b_0 = 0 \\ b_m &= -\frac{ma_{m+1}}{in}, \quad m \geq 1 \end{aligned}$$

Let $f_m = \frac{a_{m+1}}{in}$ for $m \geq 0$, $f = \sum f_m I^m$. Then $X = d_{\pi_c}f$. Hence H_n^1 is also trivial. So for $n \neq 0$ \mathfrak{X}_n is acyclic.

It follows that the Poisson cohomology of the formal neighborhood of the necklace is as in (4.10).

4.3.2 Justification for the smooth case.

To see that the cohomology of a finite small neighborhood of the necklace is the same as for the formal neighborhood we apply an argument similar to Ginzburg's [17]. For each Fourier mode consider the following exact sequence of complexes:

$$0 \rightarrow \mathfrak{X}_{n,\text{flat}}^* \rightarrow \mathfrak{X}_{n,\text{smooth}}^* \rightarrow \mathfrak{X}_{n,\text{formal}}^* \rightarrow 0$$

where $\mathfrak{X}_{n,\text{flat}}^*$ consists of smooth multivector fields whose coefficients vanish along the necklace together with all derivatives. This sequence is exact by a theorem of E. Borel. It suffices to show that the flat complex is acyclic. But $\pi_c^\# : \mathfrak{X}_{n,\text{flat}}^* \rightarrow \Omega_{n,\text{flat}}^*$ is an isomorphism since the coefficient of π_c is a polynomial in I , and every flat form can be divided by a polynomial with a flat result. Furthermore, the flat deRham complex is acyclic by the homotopy invariance of deRham cohomology.

Finally, we observe that a smooth multivector field in a neighborhood of the necklace (given by a *convergent* Fourier series) is a coboundary if and only if each mode is, and the primitives can be chosen so that the resulting series converges, as can be seen from the calculations in the previous subsection (integration can only improve convergence). Therefore, the Poisson cohomology of an annular neighborhood U of the necklace is

$$\begin{aligned} H_{\pi_c}^0(U) &= \mathbb{R} = \text{span}\{1\} \\ H_{\pi_c}^1(U) &= \mathbb{R}^2 = \text{span}\{\partial_\theta, I\partial_I\} \\ H_{\pi_c}^2(U) &= \mathbb{R} = \text{span}\{I\partial_I \wedge \partial_\theta\} \end{aligned} \tag{4.11}$$

Notice that the generators of $H_{\pi_c}^1(U)$ are the rotation $\partial_\theta = s\partial_t - t\partial_s$ (the modular vector field) and the dilation $I\partial_I = \frac{s^2+t^2-1}{2(s^2+t^2)}(s\partial_s + t\partial_t)$, while the generator of $H_{\pi_c}^2(U)$ is π_c itself, so in particular π_c does not admit rescalings even locally.

4.3.3 From local to global cohomology.

We now have all we need to compute the Poisson cohomology of a necklace Poisson structure π_c on S^2 . Cover S^2 by two open sets U and V where U is an annular neighborhood of the necklace as above, and V is the complement of the necklace consisting of two disjoint

open hemispheres on each of which π_c is nonsingular, so that the Poisson cohomology of V and $U \cap V$ is isomorphic to the deRham cohomology. The short exact Mayer-Vietoris sequence associated to this cover

$$0 \rightarrow \mathfrak{X}^*(S^2) \rightarrow \mathfrak{X}^*(U) \oplus \mathfrak{X}^*(V) \rightarrow \mathfrak{X}^*(U \cap V) \rightarrow 0$$

leads to a long exact sequence in cohomology:

$$\begin{aligned} 0 &\rightarrow H_{\pi_c}^0(S^2) \rightarrow H_{\pi_c}^0(U) \oplus H_{\pi_c}^0(V) \rightarrow H_{\pi_c}^0(U \cap V) \rightarrow \\ &\rightarrow H_{\pi_c}^1(S^2) \rightarrow H_{\pi_c}^1(U) \oplus H_{\pi_c}^1(V) \rightarrow H_{\pi_c}^1(U \cap V) \rightarrow \\ &\rightarrow H_{\pi_c}^2(S^2) \rightarrow H_{\pi_c}^2(U) \oplus H_{\pi_c}^2(V) \rightarrow H_{\pi_c}^2(U \cap V) \rightarrow 0 \end{aligned}$$

Now, the first row is clearly exact since a Casimir function on S^2 must be constant on each of the two open symplectic leaves comprising V , hence constant on all of S^2 by continuity. On the other hand, $H_{\pi_c}^1(V) = H_{\pi_c}^2(V) = H_{\pi_c}^2(U \cap V) = \{0\}$. Combining this with (4.11), we see that what we have left is

$$\begin{array}{ccccccc} & & \mathbb{R}^2 & & \mathbb{R}^2 & & \\ & & \parallel & & \parallel & & \\ 0 & \rightarrow & H_{\pi_c}^1(S^2) & \rightarrow & H_{\pi_c}^1(U) \oplus H_{\pi_c}^1(V) & \rightarrow & H_{\pi_c}^1(U \cap V) \rightarrow \\ & & & & & & \\ & & \rightarrow & H_{\pi_c}^2(S^2) & \rightarrow & H_{\pi_c}^2(U) \oplus H_{\pi_c}^2(V) & \rightarrow 0 \\ & & & \parallel & & & \\ & & & \mathbb{R} & & & \end{array}$$

Now, on the one hand, we know by Corollary 4.2.3 that $H_{\pi_c}^1(S^2)$ is at least one-dimensional; on the other hand, the restriction of the dilation vector field $I\partial_I$ to $U \cap V$ is not Hamiltonian: it corresponds under $\pi_c^\#$ to the generator of the first deRham cohomology of the annulus diagonally embedded into $U \cap V$ (a disjoint union of two annuli). It follows that $H_{\pi_c}^1(S^2)$ is exactly one-dimensional, while $H_{\pi_c}^2(S^2)$ is two-dimensional.

It only remains to identify the generators. $H_{\pi_c}^1(S^2)$ is generated by the modular class, while one of the generators of $H_{\pi_c}^2(S^2)$ is π_c itself, since its class was shown to be nontrivial even locally. The other generator is the image of $(I\partial_I, -I\partial_I) \in H_{\pi_c}^1(U \cap V)$ under the connecting homomorphism. This is somewhat unwieldy since it involves a partition of unity subordinate to the cover $\{U, V\}$ which does not yield a clear geometric interpretation of the generator. Instead, we will show directly that the standard rotationally invariant

symplectic Poisson structure π on S^2 is nontrivial in $H_{\pi_c}^2(S^2)$ and so can be taken as the second generator.

Lemma 4.3.1. *The class of the standard $SU(2)$ -invariant Poisson structure π on S^2 is nonzero in $H_{\pi_c}^2(S^2)$.*

Proof. We will work in coordinates (s, t) on the unit disk in which π and π_c are given, respectively by (4.9) and (4.8). Locally π is a coboundary whose primitive is given by an Euler vector field $E = \frac{1}{2(c-1)}(s\partial_s + t\partial_t)$: it's easy to check that $[\pi_c, E] = \pi$. But E does not extend to a vector field on S^2 since it does not behave well “at infinity”, i.e on the unit circle in the (s, t) -plane. Therefore, to prove that π is globally nontrivial it suffices to show that there does not exist a Poisson vector field X such that $E + X$ is tangent to the unit circle and the restriction is rotationally invariant. In fact, it suffices to show that there is no Hamiltonian vector field X_f such that $E + X_f$ vanishes on the unit circle (since we can always add a multiple of the modular vector field to cancel the rotation). Assuming that such an f exists, we will have, in the polar coordinates $s = r \cos \phi$, $t = r \sin \phi$:

$$E + X_f = \frac{1}{2(c-1)}r\frac{\partial}{\partial r} + \frac{1}{2r}(r^2 - \frac{1-c}{2})\left(\frac{\partial f}{\partial \phi}\frac{\partial}{\partial r} - \frac{\partial f}{\partial r}\frac{\partial}{\partial \phi}\right)$$

Upon restriction to $r = 1$ this becomes

$$(E + X_f)|_{r=1} = \left(\frac{1}{2(c-1)} + \frac{c+1}{4}\frac{\partial f}{\partial \phi}\Big|_{r=1}\right)\frac{\partial}{\partial r}\Big|_{r=1} + \frac{c+1}{4}\frac{\partial f}{\partial r}\Big|_{r=1}\frac{\partial}{\partial \phi}\Big|_{r=1}$$

In order for this to vanish it is necessary, in particular, that $\frac{\partial f}{\partial \phi}\Big|_{r=1}$ be a nonzero constant which is impossible since f is periodic in ϕ . \square

We have now arrived at our final result:

Theorem 4.3.2. *The Poisson cohomology of a necklace Poisson structure π_c on S^2 is given as follows:*

$$\begin{aligned} H_{\pi_c}^0(S^2) &= \mathbb{R} = \text{span}\{1\} \\ H_{\pi_c}^1(S^2) &= \mathbb{R} = \text{span}\{\Delta_\omega\} \\ H_{\pi_c}^2(S^2) &= \mathbb{R}^2 = \text{span}\{\pi_c, \pi\} \end{aligned}$$

Corollary 4.3.3. π_c does not admit infinitesimal rescaling.

Corollary 4.3.4. *The necklace structures π_c and $\pi_{c'}$ for $c \neq c'$ are nontrivial deformations of each other.*

Proof. $\pi_{c'} - \pi_c$ is a nonzero multiple of π but π is nontrivial in $H_{\pi_c}^2(S^2)$. \square

Appendix A

Poisson manifolds and Poisson cohomology

Definition A.0.5. A *Poisson manifold* is a manifold P together with an \mathbb{R} -bilinear skew-symmetric operation $\{\cdot, \cdot\}$ on $C^\infty(M)$, called the *Poisson bracket*, satisfying the following properties:

- The Leibniz rule: $\forall f, g, h \in C^\infty(M)$,

$$\{f, gh\} = \{f, g\}h + f\{g, h\}$$

- The Jacobi identity: $\forall f, g, h \in C^\infty(M)$,

$$\{\{f, g\}, h\} + \{\{h, f\}, g\} + \{\{g, h\}, f\} = 0$$

Since $\{\cdot, \cdot\}$ is skew-symmetric and satisfies the Leibniz rule, there exists a bivector field $\pi \in \mathfrak{X}^2(M) = \Gamma(\bigwedge^2 TM)$ such that

$$\{f, g\} = (df \wedge dg)(\pi)$$

This bivector field is called the *Poisson structure*. To express the Jacobi identity in terms of π , recall that the *Schouten bracket* of multivector fields is defined as the unique extension $[\cdot, \cdot]$ of the commutator bracket of vector fields and the action of vector fields on functions to $\mathfrak{X}^*(M) = \Gamma(\bigwedge^* TM)$ such that:

1. $[X, Y] = -(-1)^{pq}[Y, X]$, for $X \in \mathfrak{X}^{p+1}(M)$, $Y \in \mathfrak{X}^{q+1}(M)$,

2. $[X, f] = X \cdot f$ for $X \in \mathfrak{X}(M)$, $f \in C^\infty(M)$,
3. If $X, Y \in \mathfrak{X}(M)$, $[X, Y]$ is the commutator bracket,
4. For $X \in \mathfrak{X}^{p+1}(M)$, $[X, \cdot]$ is a derivation of degree p of the exterior multiplication on $\mathfrak{X}^*(M)$.

The Schouten bracket satisfies the graded Jacobi identity

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{pq}[Y, [X, Z]]$$

for $X \in \mathfrak{X}^{p+1}(M)$, $Y \in \mathfrak{X}^{q+1}(M)$, $Z \in \mathfrak{X}^{r+1}(M)$. One then checks that the bracket on $C^\infty(M)$ given by a $\pi \in \mathfrak{X}^2(M)$ satisfies the Jacobi identity if and only if π satisfies

$$[\pi, \pi] = 0 \tag{A.1}$$

A smooth map $f : (P_1, \pi_1) \rightarrow (P_2, \pi_2)$ is called a *Poisson map* if $f_*\pi_1 = \pi_2$. The standard constructions such as Poisson submanifolds and direct products are defined in an obvious manner. In terms of the Poisson brackets, a submanifold $N \subset P$ is Poisson if and only if its vanishing ideal $I_N \subset C^\infty(P)$ is a Poisson ideal; if it is merely a Poisson subalgebra, N is said to be *coisotropic*.

There are several important geometric objects associated with a Poisson bivector field π . First, it gives rise to a bundle map $\pi^\# : T^*P \rightarrow TP$ given by $\langle \alpha, \pi^\# \beta \rangle = \langle \alpha \wedge \beta, \pi \rangle$ for any $\alpha, \beta \in T_p^*P$. To any function $f \in C^\infty(P)$ one associates its *Hamiltonian vector field* X_f by

$$X_f \cdot g = \langle dg, X_f \rangle = \{g, f\} = \langle dg, \pi^\# df \rangle = [\pi, f] \cdot g$$

The image of $\pi^\#$ is a (generally singular) integrable distribution on P . Since by definition every Hamiltonian vector field is tangent to each integral submanifold, it follows easily that the integral submanifolds are Poisson submanifolds of P called the *symplectic leaves* of P because the restriction of π to each leaf is nonsingular, hence symplectic. In general the leaves have different dimensions.

A Poisson structure π also gives rise to an operator

$$d_\pi : \mathfrak{X}^* \longrightarrow \mathfrak{X}^{*+1}$$

on multivector fields given by $d_\pi(X) = [\pi, X]$. The graded Jacobi identity for the Schouten bracket combined with (A.1) implies that $d_\pi^2 = 0$, making \mathfrak{X}^* into a complex. The cohomology of this complex is called the *Poisson cohomology* of (P, π) , denoted by $H_\pi^*(P)$. The Poisson cohomology in low degrees has a clear geometric interpretation: $H_\pi^0(P)$ is the center of the Poisson algebra $C^\infty(P)$, consisting of *Casimir functions*, i.e. those whose Hamiltonian vector fields are trivial; $H_\pi^1(P)$ consists of infinitesimal Poisson automorphisms of P (Poisson vector fields) modulo inner automorphisms (Hamiltonian vector fields); $H_\pi^2(P)$ consists of nontrivial infinitesimal deformations of π and $H_\pi^3(P)$ houses obstructions to extending an infinitesimal deformation to a full deformation (see [40]).

The operator $\pi^\# : \mathfrak{X}^* \rightarrow \Omega^*$ intertwines d_π and the deRham differential d , hence induces a map $\pi^\# : H_\pi^*(P) \rightarrow H_{dR}^*(P)$ which is an isomorphism if π is symplectic. In general, however, Poisson cohomology has been notoriously difficult to compute, and there have been but a handful of successful computations ([17],[45],[18]).

For every Poisson manifold there are two Poisson cohomology classes that are special. The first one is the *modular class*, introduced by Weinstein [44]. Given a Poisson manifold (P, π) with a volume form ω , Weinstein defines an operator Δ_ω on $C^\infty(P)$ that associates to every function the divergence of its Hamiltonian vector field with respect to ω . It turns out that Δ_ω is in fact a vector field called the *modular vector field* of π with respect to ω . Moreover, Δ_ω preserves π ; if ω is replaced by another volume form, Δ_ω is changed by a Hamiltonian vector field. Thus, the class of Δ_ω in $H_\pi^1(P)$ is independent of ω ; it is called the *modular class* of (P, π) and measures the obstruction to the existence of a measure on P invariant under all Poisson automorphisms. It is zero for symplectic manifolds due to the existence of the Liouville measure.

The second special class is the class of π itself in $H_\pi^2(P)$. It is the obstruction to (infinitesimal) rescaling of π . If it vanishes, π is called *exact* and there exists a vector field X such that $L_X\pi = \pi$. It is called a *Liouville vector field*. If π is symplectic, the class of π corresponds under $\pi^\#$ to the class of the symplectic form ω in the deRham cohomology, hence it admits infinitesimal rescaling if and only if ω is exact.

Appendix B

Supermanifolds

We recall here the rudiments of the theory of supermanifolds that should suffice for the understanding of the material in the main text. For a more thorough introduction the interested reader should consult [36], [41] or [30].

B.1 Algebra

The basic setting of supermathematics is the category Super of \mathbb{Z}_2 -graded vector spaces $V = V_0 \oplus V_1$. The \mathbb{Z}_2 -grading is called *parity*. Elements of V_0 are called *even*, while elements of V_1 are *odd*; the parity of an element is denoted by a tilde over it.

If V_0 and V_1 are finite-dimensional, the dimension of V takes values in $\mathbb{Z}[\Pi]/(\Pi^2 - 1)$, the group ring of \mathbb{Z}_2 , and is denoted by $\dim V = (\dim V_0 | \dim V_1)$. The *parity reversion functor* Π is defined by

$$(\Pi V)_0 = V_1, (\Pi V)_1 = V_0$$

All the usual universal constructions, such as the direct sum, tensor product, duality and Hom carry over to the Super category, with a natural assignment of parity. The notion of an associative algebra in the Super category is the usual one, except that the multiplication must respect the parity: $V_i V_j \subset V_{i+j}$. The commutator in an associative superalgebra is the supercommutator

$$[a, b] = ab - (-1)^{\tilde{a}\tilde{b}} ba$$

One calls the algebra commutative if this bracket is identically zero.¹ In general, the sign convention - introducing $(-1)^{ij}$ whenever two symbols of parities i and j are interchanged - should be used as a matter of principle in the Super category. Thus, the notions of “symmetric” and “skew-symmetric” must be modified appropriately. An endomorphism D of V is a *derivation* if

$$D(ab) = (Da)b + (-1)^{\tilde{D}\tilde{a}}a(Db)$$

Derivations of any other kind of a bilinear operation are defined analogously.

One defines a Lie superalgebra (of parity ϵ) to be a vector space V with a bilinear skew-symmetric operation $[\cdot, \cdot]$ of parity ϵ (i.e., $[V_i, V_j] \subset V_{i+j+\epsilon}$) satisfying the Jacobi identity:

$$[a, [b, c]] = [[a, b], c] + (-1)^{(\tilde{a}+\epsilon)(\tilde{b}+\epsilon)}[b, [a, c]]$$

The parity reversion functor Π interchanges the notions of even and odd Lie superalgebras, so one can always reduce to the even case. If V also has a commutative multiplication with respect to which ad_a is a derivation (of parity $\tilde{a} + \epsilon$, of course), it becomes a Poisson superalgebra, even or odd. An odd Poisson algebra is also called a *Gerstenhaber algebra*. Various Schouten-like bracket structures encountered in the main text are Gerstenhaber algebras. It is no longer possible to reduce Gerstenhaber algebras to even Poisson algebras by parity reversion.

B.2 Affine superspaces and superdomains

A function of odd variables ξ^1, \dots, ξ^m is an element of the free commutative algebra generated by these variables, i.e. the Grassman algebra $\bigwedge(\mathbb{R}^m)^*$:

$$f(\xi^1, \dots, \xi^m) = f_0 + \xi^\mu f_\mu + \frac{1}{2}\xi^{\mu_1}\xi^{\mu_2}f_{\mu_1\mu_2} + \dots + \frac{1}{n!}\xi^{\mu_1} \dots \xi^{\mu_m}f_{\mu_1 \dots \mu_n}$$

where

$$f_{\mu_1 \dots \mu_k} = (-1)^\sigma f_{\mu_{\sigma(1)} \dots \mu_{\sigma(k)}}$$

for any permutation $\sigma \in S_k$. The variables ξ^μ are to be interpreted as coordinates on the purely odd affine superspace $\mathbb{R}^{0|m}$, which can be thought of as the result of applying the

¹ We try not to abuse the prefix “super”, omitting it whenever it is clear from the context that we are working in the Super category.

parity reversal functor Π to R^m . More invariantly, one says that the algebra of functions on the superspace ΠV is the Grassman algebra $\bigwedge V^*$.

If the coefficients in the above expression are themselves smooth functions of even variables x^1, \dots, x^n defined on \mathbb{R}^n or an open subset $U_0 \subset \mathbb{R}^n$, one says that we are given a function on an *affine superspace* $\mathbb{R}^{n|m}$ or a *superdomain* $U^{n|m} \subset \mathbb{R}^{n|m}$. These functions form a supercommutative algebra which is just the tensor product $C^\infty(U_0) \otimes \bigwedge(\mathbb{R}^m)^*$. The domain U_0 is called the *support* of $U^{n|m}$ and uniquely determines it since we cannot “bound” the odd variables. It is often convenient not to separate the even and odd variables explicitly but denote them by a collective symbol $\{x^A\}$ and assign parity to each index A .

If $(t^1, \dots, t^p, \tau^1, \dots, \tau^q)$ is another set of variables, we can define a substitution

$$\begin{aligned} x^a &= x^a(t, \tau) = x_0^a(t) + \frac{1}{2}\tau^{\alpha_1}\tau^{\alpha_2}x_{\alpha_1\alpha_2}^a(t) + \dots \\ \xi^\mu &= \xi^\mu(t, \tau) = \tau^\alpha \xi_\alpha^\mu(t) + \frac{1}{6}\tau^{\alpha_1}\tau^{\alpha_2}\tau^{\alpha_3}\xi_{\alpha_1\alpha_2\alpha_3}^\mu(t) + \dots \end{aligned} \quad (\text{B.1})$$

where $x_0^a(t), \dots$ are smooth functions defined on a domain $V_0 \subset \mathbb{R}^p$, and plug these expression into any $f = f(x, \xi)$. This is not a problem since the τ 's are nilpotent.

Example. $\sin(t + \tau^1\tau^2) = \sin t + \tau^1\tau^2 \cos t$

We interpret such substitutions as smooth maps $V^{p|q} \rightarrow U^{m|n}$. For example, the inclusion of the support $U_0 \hookrightarrow U^{n|m}$ is defined by setting all the odd variables to zero. More formally, the algebras $C^\infty(U_0) \otimes \bigwedge(\mathbb{R}^m)^*$ form a category with morphisms defined by the substitutions above, and we simply define the category of superdomains to be the opposite category. What makes the whole theory of supermanifolds nontrivial is the possibility of mixing the even and odd variables by allowing nonlinear terms in (B.1). If $V^{p|q} = U^{n|m}$ and the substitutions (B.1) are invertible, one can think of (t, τ) as giving a new coordinate system on $U^{n|m}$. Thus the domain $U^{n|m}$ is viewed as the intrinsic geometric object for which we can choose a coordinate representation at will. This is completely analogous to the ordinary, purely even case, and leads to the concept of a *supermanifold*.

The derivatives $\partial/\partial x^A$ are defined in the usual manner, as linear endomorphisms such that one has the Leibniz rule

$$\frac{\partial(fg)}{\partial x^A} = \frac{\partial f}{\partial x^A}g + (-1)^{\tilde{A}\tilde{f}}f\frac{\partial g}{\partial x^A}$$

and

$$\frac{\partial x^B}{\partial x^A} = \delta_A^B$$

Then the usual equality of mixed partials holds with appropriate signs,² and so does the inverse function theorem: the substitution (B.1) is locally invertible if and only if its Jacobian matrix is.

The only difficulty is the absence of a good notion of points: the only numerical value an odd variable can be assigned is zero, so the values that can be assigned to functions on a superdomain can only determine its support. For this reason we try to formulate all our statements in terms of the algebra of functions; whenever we mention points, we mean “running points”, i.e., all objects considered will be allowed to depend, explicitly or implicitly, on any number of even and odd parameters.

B.3 Supermanifolds

Smooth supermanifolds are glued together out of domains $U^{n|m}$ in the same way in which ordinary manifolds are glued out of coordinate domains. To make this rigorous, one has to use sheaf theory. One considers a locally ringed space $M = (M_0, \mathcal{O}_M)$ where M_0 is a topological space and \mathcal{O}_M is a sheaf of supercommutative algebras on M_0 whose stalk \mathcal{O}_x over each point $x \in M_0$ is local. A superdomain $U^{n|m} = (U_0, C^\infty(U_0) \otimes \bigwedge V^*)$ is such a space. A *chart* on M is, by definition, an isomorphism of locally ringed spaces $\phi : V = (V_0, \mathcal{O}_M|_{V_0}) \rightarrow U^{n|m}$, where $V_0 \subset M_0$ is an open subset. One says that M is a *supermanifold* if it can be covered by a countable system of charts, called an *atlas*. Atlases form a directed set under refinement, and every atlas is contained in a maximal one. Factoring $\mathcal{O}_M|_{V_0}$ by its nilradical one gets an atlas on M_0 making it a smooth manifold called, naturally, the *support* of M .

The charts ϕ give a system of local coordinates on M , allowing us to describe supermanifolds and their morphisms in coordinates while making almost no use of sheaves. Due to the absence of a good notion of points mentioned above, this is the best course to follow. In this way, all the standard constructions - products, co-products, fibered products, submanifolds, vector bundles - carry over to supermanifolds. Thus, a closed submanifold (or, more generally, a singular subvariety) is locally given by a system of equations: for example, the support M_0 is given by setting all the nilpotents to zero.

Given a supermanifold M , one defines its tangent bundle by giving, for each chart

² These are left derivatives. There are also right derivatives, satisfying the Leibniz rule when applied on the right of the argument. The difference between the left derivative of a function and a right one is only a sign. Right derivatives are not used in this work.

$(V, \{x^A\})$ on M a chart $(TV, \{x^A, \dot{x}^A\})$ on TM such that under a change of coordinates $x = x(x')$ the velocities transform in the usual way:

$$\dot{x}^A = \dot{x}^{A'} \frac{\partial x^A}{\partial x^{A'}}(x')$$

Similarly, one defines the cotangent bundle T^*M .

A basic class of examples of supermanifolds are provided by supermanifolds of the form ΠA where A is a vector bundle over an ordinary manifold M_0 . The structure sheaf of ΠA is the sheaf of smooth sections of $\bigwedge A^*$. Any atlas on M_0 gives rise to an atlas on ΠA with the coordinate transformations inherited from the vector bundle structure on A . This atlas is characterized by the property that the even coordinates transform independently of the odd ones, while the odd ones transform linearly. Such atlases are called *simple*. The fundamental classification theorem of smooth real supermanifolds (proved independently by Berezin [9], Bachelor [5] and Gawedzki [16]) asserts that any supermanifold M admits a simple atlas, i.e. globally isomorphic to one of the form ΠA . The bundle A in question is the normal bundle to the support M_0 of M , with the parity in the fibres reversed. One must emphasize, however, that this isomorphism is strictly non-canonical. For example, the cotangent bundle $T^*\Pi A$, the central object in the main text, does not possess a canonical simple atlas (Remark 3.3.3). The theorem is also false for complex analytic supermanifolds [36].

B.4 Vector fields and differential forms

A vector field on a supermanifold is simply a derivation of its algebra of functions. Vector fields on a supermanifold $M = (M_0, \mathcal{O}_M)$ form a sheaf of left \mathcal{O}_M -modules. The Jacobi-Lie bracket of vector fields is just the commutator of the corresponding derivations:

$$[X, Y]f = X(Yf) - (-1)^{\tilde{X}\tilde{Y}} Y(Xf)$$

In local coordinates, a vector field is given by an expression of the form

$$X = X^A(x) \frac{\partial}{\partial x^A} = X^a(x, \xi) \frac{\partial}{\partial x^a} + X^\mu(x, \xi) \frac{\partial}{\partial \xi^\mu}$$

Unlike the even case, if a vector field X is odd, the condition

$$[X, X] = 2X^2 = 0$$

is nontrivial. When it is satisfied, the vector field X is called *homological* because it endows the algebra of functions with the structure of a differential complex. Vector fields on M also correspond naturally to fiberwise linear functions on T^*M .

Differential forms on supermanifolds, as objects suitable for integration, are highly nontrivial [41]; however, for our purposes it suffices to consider the simplest class of differential forms on M which are polynomial functions on ΠTM . A local coordinate chart $\{x^A\}$ on M induces a chart $\{x^A, \xi^A\}$ on ΠTM where $\xi^A = dx^A$ have parity $\tilde{A} + 1$ and transform just as the notation suggests. Thus a differential form M is locally a function $\omega = \omega(x, dx)$, polynomial in dx . If this restriction is removed, we get the so-called *Bernstein-Leites pseudoforms*.

Example. $\omega = e^{-(d\xi)^2}$ is a pseudoform on $\mathbb{R}^{0|1}$.

The degree of ω as a polynomial in dx in general differs from its parity as a function on ΠTM . On ΠTM there is a canonical homological vector field, the de Rham differential

$$d = \xi^A \frac{\partial}{\partial x^A}$$

Any vector field X on M induces a vector field on ΠTM , the interior derivative

$$i_X = (-1)^{\tilde{X}} X^A \frac{\partial}{\partial \xi^A}$$

if $X = X^A \frac{\partial}{\partial x^A}$, of parity $\tilde{X} + 1$ and the Lie derivative $L_X = [d, i_X]$, of parity \tilde{X} .

B.5 Symplectic and Poisson supermanifolds

A symplectic structure on a supermanifold M is a two-form ω (i.e. a quadratic function on ΠTM),

$$\omega = \frac{1}{2} dx^A dx^B \omega_{AB}(x) = \frac{1}{2} dx^a dx^b \omega_{ab} + dx^a d\xi^\mu \omega_{a\mu} + \frac{1}{2} d\xi^\mu d\xi^\nu \omega_{\mu\nu}$$

which is closed ($d\omega = 0$) and nondegenerate (the matrix ω_{AB} is invertible). One distinguishes even and odd symplectic supermanifolds, depending on the parity of ω . In the even case, the nondegeneracy of ω is equivalent to the invertibility of the real matrices ω_{ab} and $\omega_{\mu\nu}$, after setting the nilpotents to zero. Note that ω_{ab} is skew-symmetric while $\omega_{\mu\nu}$ is symmetric; the dimension of M in this case has to be $2n|m$, and the signature of $\omega_{\mu\nu}$ is an invariant. In the odd case, the nondegeneracy is equivalent to the invertibility of the real matrix $\omega_{a\mu}$,

hence the dimension of M must be $n|n$. The Darboux theorem holds for supermanifolds and asserts that ω locally has the standard form

$$\omega = dp_a dq^a + \frac{1}{2} \sum_{\mu} \pm (d\gamma^{\mu})^2$$

for $\tilde{\omega} = 0$, and

$$\omega = d\theta_a dx^a$$

for $\tilde{\omega} = 1$. Even and odd symplectic supermanifolds have very different properties.

For differential forms there is a natural notion of pullback, in particular, restriction to submanifolds. As usual, one calls a submanifold L of M *Lagrangian* if the restriction of ω to L is identically zero and L is of the maximal dimension where it is possible. Lagrangian submanifolds of an even symplectic supermanifold $M^{2n|m}$ have dimension $n|[m/2]$ (they may not even exist if the signature of $\omega_{\mu\nu}$ is nonzero), while those of an odd one $M^{n|n}$ have dimension $k|n - k$.

Given a function f on M , its hamiltonian vector field is defined by the formula

$$i_{X_f} \omega = -df$$

and for a pair of functions f, g their Poisson bracket is defined by

$$\{f, g\} = X_f g$$

If $\tilde{\omega} = 0$, $(C^\infty(M), \{\cdot, \cdot\})$ becomes an even Poisson algebra; if $\tilde{\omega} = 1$, a Gerstenhaber algebra.

A typical example of an even symplectic supermanifold is T^*Q , where Q is a supermanifold, with the standard symplectic structure

$$\omega = dx_A^* dx^A$$

(the momenta x_A^* have the same parity as x^A), whereas a typical odd symplectic supermanifold is ΠT^*Q with

$$\omega = d\theta_A dx^A$$

($\tilde{\theta}_A = \tilde{x}^A + 1$). Functions on ΠT^*Q coincide with multivector fields on Q , and the canonical odd Poisson bracket is nothing but the Schouten bracket of multivector fields.

One can similarly introduce even or odd Poisson supermanifolds that are not necessarily symplectic. For example, any Lie algebroid A gives rise to an odd Poisson structure on ΠA^* via the generalized Schouten bracket, as explained in the main text.

Remark. Finally, we remark that we have left out the part of the supermanifold theory that is perhaps the most interesting and different from the ordinary manifold case - the integration theory. This is only because it is not used anywhere in the main body of this work. The interested reader is strongly advised to look in the treatise [41] for a thorough exposition.

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